

Single and multiple material constraints in thermoelasticity

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Abstract: Constraints on the possible forms of material response, such as incompressibility or inextensibility, have long been used to simplify constitutive response models, and have resulted in substantial progress in fields such as fluid mechanics and the mechanics of composite materials. A method of imposing these constraints for thermoelastic materials is considered that follows steps that remove the need for assuming an additive term resulting from the constraint. In the process, three methods are considered for the separation of the constitutively prescribed part of the response from the part that is in reaction to the constraints. Both the case of single and multiple constraints are considered with extensive examples including special considerations for including effects associated with isotropic or anisotropic thermal expansion.

Key Words: Thermoelastic, material constraint, internal constraint, geometric constraint, multiple constraints, thermodynamics, nonlinear elasticity, incompressibility, inextensibility, Bell constraint, isotropic constraints, anisotropic thermal expansion

1. INTRODUCTION

The theory of material response in the presence of constraints has been looked at by many authors and has an important place in the development of material response models. Internal constraints may take many different forms and may be applied in many different situations. Possibly, the most commonly used and studied material constraints are the geometric constraints of incompressibility and inextensibility. These and other constraints are idealizations representing the difficulty to initiate certain events relative to others. For example, the relative rigidity of the fiber compared to the matrix in a fiber reinforced composite can be modeled by assuming the matrix is inextensible along the fiber directions, such as is done by Spencer [1]. Another application of material constraints is in the development of theories for problems that are characterized by specific global geometric characteristics of the deformed body, as was done, for example, in the development of the theory of rods and shells by Antman and Marlow [2]. The consideration of constraints is not limited to geometric constraints, as the constraint is simply an imposed relation between the variables of a theory. For example, in Negahban [3] the influence of the yield function in plasticity is introduced as a material constraint.

The effect of each constraint is normally twofold. A kinematical constraint results in a reduction in the number of degrees of freedom associated with the deformation and, also, simultaneously results in the introduction of the possibility of having certain changes in stress without any change in the deformation. For example, irrespective of loading, the length of material line elements along a direction of inextensibility do not change. As such, the constraint has reduced the degrees of freedom associated with the deformation (i.e., certain changes in the shape of the material cannot occur). In addition, it is obvious that loads imposed on a material element along a direction of inextensibility cannot change the shape, yet alter the traction and, therefore, change the stress. This implies that stress is not only a function of the deformation, but that additional information is needed to fully evaluate it.

In a mechanical theory, the primary question is how the constraint influences the evaluation of stress. That is, given a deformation, what can be said about the stress, what remains unknown about the stress until additional information is provided, and what additional information is needed to fully evaluate the stress. If one does not restrict attention to mechanical theories and geometric constraints, one must ask what variables are affected by the introduction of the constraint, in addition to how each variable is influenced.

The consideration of internal constraints has followed several paths depending on the starting point of the construction and the domain of application. For purely mechanical constraints, such as incompressibility and inextensibility in elasticity, Ericksen and Rivlin [4] used the balance of energy and the method of Lagrange multipliers to obtain the appropriate expression for stress. This method was also used by Green and Adkins [6]. Carlson and Tortorelli [5] use a similar method replacing the method of Lagrange multipliers by a geometrically motivated argument. An alternate line of development taken by many authors starts by assuming that the constraint is “workless”, as was done by Truesdell and Noll [7], and others [8, 9, 10, 11, 12]. For thermoelastic materials, most of the traditional developments are based on imposing the second law of thermodynamics in the form of the entropy production inequality or the Clausius–Duhem inequality, such as was done by Rivlin [13], Green and Zerna [14], Green et al. [15], Green and Naghdi [16], and Gurtin and Podio-Guidugli [17]. A notable exception is the work of Batra [18] that uses Hamilton’s extended principle to obtain the effects of constraints, and the work of Casey and Krishnaswamy [19] that separate the second law into two parts following a procedure suggested by Rivlin. In addition to the above, references to work on constraints in elasticity can be found in the recent work of Martins and Duda [20], Klisch [21], and Podio-Guidugli [22]. A historical discussion of the work done on constraints is presented at the end of this article.

In the following presentation it is shown how one can impose internal material constraints starting from the Clausius–Duhem inequality, without introducing additional assumptions, such as that of assuming, a priori, an additive decomposition of terms into a constitutively prescribed part and a part due to the constraint. The method is based on directly including parameters associated with the resulting indeterminacies into the constitutive models and then using the Clausius–Duhem inequality to obtain the resulting relations. These added parameters may be of scalar or tensor form, yet the indeterminacy in the constitutive equations resulting from scalar constraints turn out to be of scalar form. It is shown that there are multiple ways to decompose the response (stress and entropy) into a part that is fully determined from the deformation and temperature (extra stress and extra entropy), and one that is in reaction to the constraint (reaction stress and reaction entropy).

Three methods to group terms are considered. Each method makes the two parts normal to each other using a different definition for normality. The results are obtained for both single and multiple material constrains. Examples are presented for each case including single and multiple constrains. The single constraints considered are isothermally incompressible, isothermally inextensible, isothermally Bell constrained with anisotropic thermal expansion, a general isothermal constraint with anisotropic thermal expansion, a general isothermal isotropic constraint, and the constraint of zero temperature gradient perpendicular to a material surface. The multiple constraints considered are isothermally incompressible and inextensible, isothermally incompressible and Bell constrained, isothermally inextensible along two directions, and isothermally inextensible along three directions. The Appendix includes the examples of isothermally constrained trace of the right Cauchy stretch, isothermally Bell constrained with isotropic thermal expansion, isothermally inextensible and Bell constrained, and isothermally Bell constrained and inextensible along two directions.

2. NOTATION

As is common in most descriptions of continuum mechanics, “ \mathbf{X} ” will denote the position of material particles in a reference configuration, “ \mathbf{x} ” will denote the position in the current configuration, and “ \mathbf{F} ” will denote the deformation gradient described by $d\mathbf{x} = \mathbf{F}d\mathbf{X}$. The temperature will be denoted by “ θ ,” its gradient with respect to changes in the current configuration will be denoted by “ \mathbf{g} ” and its gradient with respect to changes in the reference configuration will be denoted by “ \mathbf{G} ” so that one has $d\theta = \mathbf{g} \circ d\mathbf{x} = \mathbf{G} \circ d\mathbf{X}$, where “ \circ ” denotes the dot product. The velocity gradient will be denoted by “ \mathbf{L} ” and is defined as $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$, where the “ $\dot{}$ ” in $\dot{\mathbf{F}}$ denotes the material time derivative. A superscript “ \mathbf{T} ” is used to designate the transpose.

Four response functions will be considered in this development. These four are the response functions for the specific free-energy, ψ , specific entropy, η , Cauchy stress, \mathbf{T} , and the heat flux vector, \mathbf{q} . The term “response” in this development denotes a dependent variable for which one must provide a constitutive model (function) in terms of the independent variables of the theory (the distinction between independent and dependent variables is not necessarily unique, as, for example, sometimes temperature is replaced for entropy as a dependent variable [2,8]). The independent variables in unconstrained thermoelasticity are normally taken to be position in the reference configuration, \mathbf{X} , to allow for variation of material characteristics from point to point in the material body, deformation gradient, \mathbf{F} , to characterize the geometric distortion of the body, temperature, θ , and its variation in space described by either \mathbf{G} or \mathbf{g} .

3. UNCONSTRAINED THERMOELASTIC MATERIAL

An unconstrained thermoelastic material will be defined as a material for which the current response at each material point can be modeled by functions of the current values of the

deformation gradient, temperature, and temperature gradient. This provides a fairly general starting point from which one can develop most every possible model. The constitutive functions under this assumption can be written as

$$\psi = \psi^\dagger(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}), \quad (1)$$

$$\eta = \eta^\dagger(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}), \quad (2)$$

$$\mathbf{T} = \mathbf{T}^\dagger(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}), \quad (3)$$

$$\mathbf{q} = \mathbf{q}^\dagger(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}), \quad (4)$$

where in each equation the left-hand side is the response, and the right hand side is the constitutive function used to model the response, with the superscript “ \dagger ” used to designate the response function.

The second law of thermodynamics, written in the form known as the Clausius–Duhem inequality (see [2 or 3]), is given as

$$\rho \dot{\psi} - \text{tr}(\mathbf{T}\mathbf{L}) + \rho \eta \dot{\theta} + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0, \quad (5)$$

where ρ is density, and “ tr ” denotes the trace. This inequality, which is the manifestation of the assumption that entropy in a body increases at a rate equal to or greater than the entropy added to the body by heat, is used in this development to eliminate from consideration material response functions (constitutive models) that can result in response inconsistent with the second law of thermodynamics.

The form of the constitutive model for specific free-energy given in (1) results in a material time derivative expressed as

$$\dot{\psi} = \partial_{\mathbf{F}}(\psi) : \dot{\mathbf{F}} + \partial_{\theta}(\psi) \dot{\theta} + \partial_{\mathbf{G}}(\psi) \circ \dot{\mathbf{G}}, \quad (6)$$

where $\partial_{\mathbf{A}}(\psi)$ denotes the partial derivative of ψ with respect to \mathbf{A} (for example, $\partial_{\mathbf{A}}(\psi) = \frac{\partial \psi}{\partial A_{ij}} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$ for second order tensor $\mathbf{A} = A_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$ given in the fixed orthonormal base $\hat{\mathbf{e}}_i$, where “ \otimes ” is the tensor product, also known as the dyadic product), $\mathbf{A} : \mathbf{B} = \text{tr}(\mathbf{A}^T \mathbf{B})$ for second order tensors \mathbf{A} and \mathbf{B} ($\mathbf{A} : \mathbf{B} = A_{ij} B_{ij}$ for $\mathbf{A} = A_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$ and $\mathbf{B} = B_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$ for orthonormal base vectors $\hat{\mathbf{e}}_i$). Using the relations $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$ and $\text{tr}(\mathbf{A}\mathbf{B}) = \mathbf{A}^T : \mathbf{B}$, one can show that

$$\text{tr}(\mathbf{T}\mathbf{L}) = \text{tr}(\mathbf{T}\dot{\mathbf{F}}\mathbf{F}^{-1}) = \text{tr}(\mathbf{F}^{-1}\mathbf{T}\dot{\mathbf{F}}) = (\mathbf{F}^{-1}\mathbf{T})^T : \dot{\mathbf{F}}. \quad (7)$$

Using this expression for $\text{tr}(\mathbf{T}\mathbf{L})$, substitution of $\dot{\psi}$ from (6) into the Clausius–Duhem inequality (5), and reorganization results in

$$[\rho \partial_{\mathbf{F}}(\psi) - (\mathbf{F}^{-1}\mathbf{T})^T] : \dot{\mathbf{F}} + \rho [\partial_{\theta}(\psi) + \eta] \dot{\theta} + [\partial_{\mathbf{G}}(\psi)] \circ \dot{\mathbf{G}} + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0. \quad (8)$$

This inequality should hold for all possible loading processes. Since the material is in no way constrained, at each loading point described by $\mathcal{L} = (\mathbf{F}, \theta, \mathbf{G})$, the loading rate described by $\dot{\mathcal{L}} = (\dot{\mathbf{F}}, \dot{\theta}, \dot{\mathbf{G}})$ can take any arbitrary value independent of the current value of \mathcal{L} , and each component of $\dot{\mathcal{L}}$ can be selected independent of its other components. Since the terms in the three square brackets and the last term are each only a function of \mathcal{L} , and since the components of $\dot{\mathcal{L}}$ can each take any arbitrary values independent of \mathcal{L} and the other components of $\dot{\mathcal{L}}$, then, as a result of the following lemma, one can conclude that the three terms in the square brackets must each be equal to zero and the last term must be less-than or equal to zero if the inequality is to be satisfied under all conditions.

Lemma 1. Consider the inequality

$$\sum_{i=1,n} \mathcal{G}_i(\alpha_1, \dots, \alpha_m) \varpi_i + \mathcal{G}_{n+1}(\alpha_1, \dots, \alpha_m) \leq 0, \quad (9)$$

which must hold for all admissible α_i and for all ϖ_i , where each \mathcal{G}_i is an arbitrary bounded function of the variable set $(\alpha_1, \dots, \alpha_m)$. If the variables α_i and ϖ_i are independent, and each ϖ_i can be selected independent of the other ϖ_i , then the inequality can only be satisfied if

$$\mathcal{G}_i(\alpha_1, \dots, \alpha_m) = 0 \quad \forall i = 1, \dots, n \quad (10)$$

and

$$\mathcal{G}_{n+1}(\alpha_1, \dots, \alpha_m) \leq 0. \quad (11)$$

The proof of the lemma is fairly easy and is included in the Appendix. The key to the use of this lemma is the establishment of the independence of functions $\mathcal{G}_i(\alpha_1, \dots, \alpha_m)$ from the selection of ϖ_i , the establishment of the independence of the ϖ_i from each other, and the determination that each ϖ_i may be selected arbitrarily.

In the case of the inequality given in (8), the conditions needed to invoke the lemma are satisfied for each set of loading parameters \mathcal{L} that result in nonsingular response functions. As stated above, each term in the square brackets must be equal to zero, and the last term on the left-hand side of (8) must be less than or equal to zero resulting in the relations

$$\mathbf{T}^T = \rho \partial_{\mathbf{F}}(\psi) \mathbf{F}^T, \quad (12)$$

$$\eta = -\partial_{\theta}(\psi), \quad (13)$$

$$\partial_{\mathbf{G}}(\psi) = 0, \quad (14)$$

$$\frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0. \quad (15)$$

The first and second equations in this set relate the Cauchy stress and specific entropy to derivatives of the specific free-energy, the third establishes that the specific free-energy can never depend on temperature gradient, and the inequality states that the heat flux vector can

never have a positive projection along the direction of the temperature gradient, yielding the physical understanding that heat always flows from hot to cold. As can be seen, enforcement of the Clausius–Duhem inequality has removed from consideration all models for which free-energy, and hence the Cauchy stress and entropy, depend on temperature gradient, only allowing heat flux to depend on this variable. Embedded in the development of these results is the assumption that temperature and density are always strictly positive, and density, through the law of conservation of mass, is given in terms of the density in the reference configuration, ρ_o , by the equation $\rho \det(\mathbf{F}) = \rho_o$, clearly establishing it as a function of \mathcal{L} and \mathbf{X} .

4. A SINGLE MATERIAL CONSTRAINT

A material constraint is a relation between the thirteen components of the loading $\mathcal{L} = (\mathbf{F}, \theta, \mathbf{G})$. For example, the constraint equation describing incompressibility is

$$\det(\mathbf{F}) = \text{constant}, \quad (16)$$

where $J = \det(\mathbf{F})$ is the volume ratio relative to the volume in the reference configuration. The constant is unity if the reference configuration is taken to be a configuration that the material actually takes, such as the initial configuration.

A general constraint is given by scalar function f of the form $f(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}) = 0$. For each material point, f represents a relation between the deformation gradient, temperature, and temperature gradient. This can be written in terms of the loading $\mathcal{L} = (\mathbf{F}, \theta, \mathbf{G})$ as

$$f(\mathbf{X}, \mathcal{L}) = 0. \quad (17)$$

As shown schematically in Figure 1, the constraint represents a surface in the 13-dimensional space of \mathcal{L} . Even though one can have several simultaneous constraint conditions, focus will be first put on a single scalar constraint. If more than one constraint condition exists, they need to be compatible in the sense that satisfying one constraint will not exclude the possibility of satisfying the others. Such issues will be addressed in the next section.

The existence of a material constraint also changes the characteristics of the constitutive response functions. For example, as mentioned in the introduction, the addition of load along the direction of inextensibility will not result in any changes in shape and, therefore, will not result in changes of the deformation gradient, even though it will increase the traction. The stress, defined in terms of the traction, therefore can change without any change in the deformation. As a result, the stress is no longer fully determined by the knowledge of $(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G})$. Let p be a scalar¹ that provides the additional information needed to calculate the response (i.e., that information needed, in addition to \mathbf{X} and \mathcal{L} , to be able to completely evaluate the response). It will be assumed that all constitutive functions depend on this additional variable.² That is,

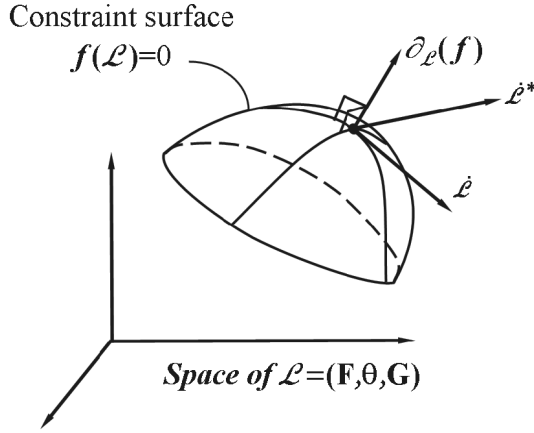


Figure 1. The schematic of the 13-dimensional loading space \mathcal{L} , the 12-dimensional surface described by the constraint $f(\mathbf{X}, \mathcal{L}) = 0$, an arbitrary loading rate $\dot{\mathcal{L}}^*$, a loading rate $\dot{\mathcal{L}}$ tangent to the loading surface and, therefore, consistent with the constraint, and the normality of the gradient of f to the constraint surface.

$$\begin{aligned}
 \psi &= \psi^\dagger(\mathbf{X}, \mathcal{L}, p), \\
 \mathbf{T} &= \mathbf{T}^\dagger(\mathbf{X}, \mathcal{L}, p), \\
 \eta &= \eta^\dagger(\mathbf{X}, \mathcal{L}, p), \\
 \mathbf{q} &= \mathbf{q}^\dagger(\mathbf{X}, \mathcal{L}, p).
 \end{aligned} \tag{18}$$

The constraint restricts how the components of \mathcal{L} can change. The relation between the rates of change of these components with respect to time can be obtained by taking the material time derivative of the constraint to obtain

$$\partial_{\mathbf{F}}(f) : \dot{\mathbf{F}} + \partial_{\theta}(f)\dot{\theta} + \partial_{\mathbf{G}}(f) \circ \dot{\mathbf{G}} = 0. \tag{19}$$

As can be seen, this is a scalar relation between the thirteen component of $\dot{\mathcal{L}} = (\dot{\mathbf{F}}, \dot{\theta}, \dot{\mathbf{G}})$, and thus reduces the degrees of freedom of $\dot{\mathcal{L}}$ from thirteen to twelve. As a result, one can no longer arbitrarily assign values to all thirteen components of $\dot{\mathcal{L}}$. To simplify the presentation this equation will be written as

$$\partial_{\mathcal{L}}(f) \circ \dot{\mathcal{L}} = 0, \tag{20}$$

where “ \circ ” denotes the general inner product defined by (19). The constraint condition $\partial_{\mathcal{L}}(f) \circ \dot{\mathcal{L}} = 0$ states that all admissible loading rates $\dot{\mathcal{L}}$ are “orthogonal” to $\partial_{\mathcal{L}}(f)$, as shown schematically in Figure 1. That is, the projection of $\dot{\mathcal{L}}$ onto $\partial_{\mathcal{L}}(f)$ is zero. Therefore, from every *arbitrary loading rate* $\dot{\mathcal{L}}^*$ one can construct an admissible loading rate $\dot{\mathcal{L}}$ by removing the portion of $\dot{\mathcal{L}}^*$ that provides a non-zero projection onto $\partial_{\mathcal{L}}(f)$. On the other

hand, one can construct every arbitrary loading rate $\dot{\mathcal{L}}^*$ by adding to an admissible loading rate $\dot{\mathcal{L}}$ an appropriate loading rate along $\partial_{\mathcal{L}}(f)$. The latter can be written as

$$\dot{\mathcal{L}}^* = \dot{\mathcal{L}} + \alpha \partial_{\mathcal{L}}(f), \quad (21)$$

where α is a scalar factor which may be changed as needed. Using this relation, one can construct an admissible loading rate $\dot{\mathcal{L}}$ from any arbitrary loading rate $\dot{\mathcal{L}}^*$ by selecting α such that $\dot{\mathcal{L}} = \dot{\mathcal{L}}^* - \alpha \partial_{\mathcal{L}}(f)$ satisfies the constraint condition $\partial_{\mathcal{L}}(f) \circ \dot{\mathcal{L}} = 0$. This results in

$$\partial_{\mathcal{L}}(f) \circ [\dot{\mathcal{L}}^* - \alpha \partial_{\mathcal{L}}(f)] = 0, \quad (22)$$

and gives α as

$$\alpha = \frac{\partial_{\mathcal{L}}(f) \circ \dot{\mathcal{L}}^*}{\partial_{\mathcal{L}}(f) \circ \partial_{\mathcal{L}}(f)}. \quad (23)$$

Consideration of the case where $\partial_{\mathcal{L}}(f) \circ \partial_{\mathcal{L}}(f) = 0$ is not of any interest since that would imply that all the thirteen component of $\partial_{\mathcal{L}}(f)$ are zero, excluding the dependence of f on \mathcal{L} . It follows that

$$\dot{\mathcal{L}} = \dot{\mathcal{L}}^* - \frac{\partial_{\mathcal{L}}(f) \circ \dot{\mathcal{L}}^*}{\partial_{\mathcal{L}}(f) \circ \partial_{\mathcal{L}}(f)} \partial_{\mathcal{L}}(f), \quad (24)$$

which yields the relations

$$\dot{\mathbf{F}} = \dot{\mathbf{F}}^* - \frac{\partial_{\mathcal{L}}(f) \circ \dot{\mathcal{L}}^*}{\partial_{\mathcal{L}}(f) \circ \partial_{\mathcal{L}}(f)} \partial_{\mathbf{F}}(f), \quad (25)$$

$$\dot{\theta} = \dot{\theta}^* - \frac{\partial_{\mathcal{L}}(f) \circ \dot{\mathcal{L}}^*}{\partial_{\mathcal{L}}(f) \circ \partial_{\mathcal{L}}(f)} \partial_{\theta}(f), \quad (26)$$

$$\dot{\mathbf{G}} = \dot{\mathbf{G}}^* - \frac{\partial_{\mathcal{L}}(f) \circ \dot{\mathcal{L}}^*}{\partial_{\mathcal{L}}(f) \circ \partial_{\mathcal{L}}(f)} \partial_{\mathbf{G}}(f). \quad (27)$$

For the assumed form of the specific free-energy given in (18), the material time derivative is given by³

$$\dot{\psi} = \partial_{\mathcal{L}}(\psi) \circ \dot{\mathcal{L}} + \partial_p(\psi) \dot{p}, \quad (28)$$

where $\dot{\mathcal{L}}$ is restricted to loading paths that are consistent with the constraint condition (20). One can write $\dot{\psi}$ in terms of an arbitrary loading rate $\dot{\mathcal{L}}^*$ using (24) to get

$$\begin{aligned}
 \dot{\psi} &= \partial_{\mathcal{L}}(\psi) \circ \left[\dot{\mathcal{L}}^* - \frac{\partial_{\mathcal{L}}(f) \circ \dot{\mathcal{L}}^*}{\partial_{\mathcal{L}}(f) \circ \partial_{\mathcal{L}}(f)} \partial_{\mathcal{L}}(f) \right] + \partial_p(\psi) \dot{p} \\
 &= \left[\partial_{\mathcal{L}}(\psi) - \frac{\partial_{\mathcal{L}}(\psi) \circ \partial_{\mathcal{L}}(f)}{\partial_{\mathcal{L}}(f) \circ \partial_{\mathcal{L}}(f)} \partial_{\mathcal{L}}(f) \right] \circ \dot{\mathcal{L}}^* + \partial_p(\psi) \dot{p}. \tag{29}
 \end{aligned}$$

The Clausius–Duhem inequality must hold for admissible \mathcal{L} and $\dot{\mathcal{L}}$. Let us introduce into this expression the relation for $\dot{\psi}$ given in (29), and the expressions for $\dot{\mathbf{F}}$ and $\dot{\theta}$ given in (25) and (26) to get

$$\begin{aligned}
 &\rho \left[\partial_{\mathcal{L}}(\psi) - \frac{\partial_{\mathcal{L}}(\psi) \circ \partial_{\mathcal{L}}(f)}{\partial_{\mathcal{L}}(f) \circ \partial_{\mathcal{L}}(f)} \partial_{\mathcal{L}}(f) \right] \circ \dot{\mathcal{L}}^* + \rho \partial_p(\psi) \dot{p} \\
 &- (\mathbf{T}^T \mathbf{F}^{-T}) : \left[\dot{\mathbf{F}}^* - \frac{\partial_{\mathcal{L}}(f) \circ \dot{\mathcal{L}}^*}{\partial_{\mathcal{L}}(f) \circ \partial_{\mathcal{L}}(f)} \partial_{\mathbf{F}}(f) \right] \\
 &+ \rho \eta \left[\dot{\theta}^* - \frac{\partial_{\mathcal{L}}(f) \circ \dot{\mathcal{L}}^*}{\partial_{\mathcal{L}}(f) \circ \partial_{\mathcal{L}}(f)} \partial_{\theta}(f) \right] + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0. \tag{30}
 \end{aligned}$$

Reorganization of the terms yields

$$\begin{aligned}
 &\left\{ \rho \left[\partial_{\mathcal{L}}(\psi) - \frac{\partial_{\mathcal{L}}(\psi) \circ \partial_{\mathcal{L}}(f)}{\partial_{\mathcal{L}}(f) \circ \partial_{\mathcal{L}}(f)} \partial_{\mathcal{L}}(f) \right] + \frac{(\mathbf{T}^T \mathbf{F}^{-T}) : \partial_{\mathbf{F}}(f)}{\partial_{\mathcal{L}}(f) \circ \partial_{\mathcal{L}}(f)} \partial_{\mathcal{L}}(f) \right. \\
 &- \left. \rho \eta \frac{\partial_{\theta}(f)}{\partial_{\mathcal{L}}(f) \circ \partial_{\mathcal{L}}(f)} \partial_{\mathcal{L}}(f) \right\} \circ \dot{\mathcal{L}}^* - (\mathbf{T}^T \mathbf{F}^{-T}) : \dot{\mathbf{F}}^* \\
 &+ \rho \eta \dot{\theta}^* + \rho \partial_p(\psi) \dot{p} + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0, \tag{31}
 \end{aligned}$$

which must hold for every arbitrary $\dot{\mathcal{L}}^* = (\dot{\mathbf{F}}^*, \dot{\theta}^*, \dot{\mathbf{G}}^*)$, and any arbitrary \dot{p} . The reader will note that the system is linear in $\dot{\mathcal{L}}^*$ and \dot{p} , so that one can organize the equation into five terms, where the first term only contains $\dot{\mathbf{F}}^*$, the second term only contains $\dot{\theta}^*$, the third term only contains $\dot{\mathbf{G}}^*$, the fourth term only contains \dot{p} , and the fifth term is $(\mathbf{q} \circ \mathbf{g})/\theta$. In the reorganized equation, the factor multiplying each rate is a function of \mathbf{X} , \mathcal{L} and p , and independent of $\dot{\mathcal{L}}^*$ or \dot{p} . Since the conditions of the lemma are satisfied, for the equation to hold for all arbitrary values of $\dot{\mathcal{L}}^*$ and \dot{p} , the factors multiplying $\dot{\mathbf{F}}^*$, $\dot{\theta}^*$, $\dot{\mathbf{G}}^*$, and \dot{p} must each be zero and the last term on the left-hand side must be less than or equal to zero. The result of this process is the following five relations:

$$\rho \partial_{\mathbf{F}}(\psi) + \varpi \partial_{\mathbf{F}}(f) - \mathbf{T}^T \mathbf{F}^{-T} = 0, \tag{32}$$

$$\rho \partial_{\theta}(\psi) + \varpi \partial_{\theta}(f) + \rho \eta = 0, \tag{33}$$

$$\rho \partial_{\mathbf{G}}(\psi) + \varpi \partial_{\mathbf{G}}(f) = 0, \tag{34}$$

$$\partial_p(\psi) = 0, \tag{35}$$

$$\frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0, \quad (36)$$

where

$$\varpi = -\frac{\rho \partial_{\mathcal{L}}(\psi) \circ \partial_{\mathcal{L}}(f) - (\mathbf{T}^T \mathbf{F}^{-T}) : \partial_{\mathbf{F}}(f) + \rho \eta \partial_{\theta}(f)}{\partial_{\mathcal{L}}(f) \circ \partial_{\mathcal{L}}(f)}. \quad (37)$$

To simplify the presentation and bring more physical meaning to the terms, from this point on \mathbf{T}_E will be used to denote the “extra stress”, and η_E will be used to denote the “extra entropy”. These two terms are defined by

$$\mathbf{T}_E^T = \rho \partial_{\mathbf{F}}(\psi) \mathbf{F}^T, \quad \eta_E = -\partial_{\theta}(\psi) \quad (38)$$

and are the same expressions for evaluating stress and entropy when there is no constraint present. Using these expressions and reorganization of terms will yield the following five relations:

$$\mathbf{T}^T = \mathbf{T}_E^T + \varpi \partial_{\mathbf{F}}(f) \mathbf{F}^T, \quad (39)$$

$$\eta = \eta_E - \frac{\varpi}{\rho} \partial_{\theta}(f), \quad (40)$$

$$\partial_{\mathbf{G}}(\psi) = -\frac{\varpi}{\rho} \partial_{\mathbf{G}}(f), \quad (41)$$

$$\partial_p(\psi) = 0, \quad (42)$$

$$\frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0, \quad (43)$$

which must be augmented by the constraint condition $f(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}) = 0$. The following conclusions can be drawn from (39)–(43):

1. It directly follows from (42) that free-energy is independent of p .
2. The function ϖ is the only term on the right-hand side of (39)–(41) that may depend on p since density ρ , the constraint function f , and free-energy ψ are all independent of p . As is shown in (37), ϖ can inherit a dependence on p through the Cauchy stress and/or through entropy.
3. The dependence of Cauchy stress \mathbf{T} and entropy η on p only comes through the scalar function ϖ . This follows from the previous comment and Equations (39) and (40).
4. For constraint functions of the form $f(\mathbf{X}, \mathbf{G})$ that exclusively restrain the temperature gradient, it can be shown that ϖ , \mathbf{T} and η are all independent of p . The independence of ϖ from p follows from the fact that for this type of constraint $\partial_{\mathbf{F}}(f) = 0$ and $\partial_{\theta}(f) = 0$, so that from (37) one can conclude that ϖ only depends on f , ψ , and ρ , all three of which are independent of p . The independence of \mathbf{T} and η from p then directly follows from Equations (39) and (40) since none of the terms on the right-hand side can depend of p .

5. For cases that ϖ depends on p , it follows from (41) that the constraint equation should be independent of \mathbf{G} . In (41) since one knows that ψ , ρ , and f are all independent of p , it follows that $\partial_{\mathbf{G}}(f)$ must be zero (i.e., $f(\mathbf{X}, \mathbf{F}, \theta)$).
6. In cases where ϖ depends on p , it follows from the last comment that the free-energy ψ is independent of temperature gradient \mathbf{G} .
7. Either one of the Equations (39)–(41) may be used to evaluate ϖ and, therefore, one can construct any number of methods to calculate ϖ including, for example,

$$\begin{aligned}\varpi &= \frac{[(\mathbf{T}^T - \mathbf{T}_E^T)\mathbf{F}^{-T}] : \partial_{\mathbf{F}}(f)}{\partial_{\mathbf{F}}(f) : \partial_{\mathbf{F}}(f)} = \frac{[(\mathbf{T}^T - \mathbf{T}_E^T)] : [\partial_{\mathbf{F}}(f)\mathbf{F}^T]}{[\partial_{\mathbf{F}}(f)\mathbf{F}^T] : [\partial_{\mathbf{F}}(f)\mathbf{F}^T]} \\ &= \frac{\rho(\eta_E - \eta)\partial_{\theta}(f)}{\partial_{\theta}(f)\partial_{\theta}(f)} = -\frac{\partial_{\mathbf{G}}(\psi) \circ \partial_{\mathbf{G}}(f)}{\partial_{\mathbf{G}}(f) \circ \partial_{\mathbf{G}}(f)},\end{aligned}\quad (44)$$

or substitution of (41) into (37) will result in the expression for ϖ given by

$$\varpi = \frac{[(\mathbf{T}^T - \mathbf{T}_E^T)\mathbf{F}^{-T}] : \partial_{\mathbf{F}}(f) + \rho(\eta_E - \eta)\partial_{\theta}(f)}{\partial_{\mathbf{F}}(f) : \partial_{\mathbf{F}}(f) + \partial_{\theta}(f)\partial_{\theta}(f)}.\quad (45)$$

Obviously, the use of each expression is contingent on having a nonzero denominator.

The relation of ϖ to the physical constraint comes from the study of the response of the material. For example, consider an *incompressible* material given by the constraint $f = \det(\mathbf{F}) - 1 = 0$. For this constraint $\dot{f} = \det(\mathbf{F})\mathbf{F}^{-T} : \dot{\mathbf{F}}$ so that one has

$$\partial_{\mathbf{F}}(f) = \det(\mathbf{F})\mathbf{F}^{-T} = \mathbf{F}^{-T}, \quad \partial_{\theta}(f) = \partial_{\mathbf{G}}(f) = 0.\quad (46)$$

The response of such a material is therefore given by

$$\mathbf{T}^T = \mathbf{T}_E^T + \varpi\mathbf{I}, \quad \eta = \eta_E.\quad (47)$$

The first equation states that the expression for Cauchy stress derived without consideration of the constraint (\mathbf{T}_E) falls short of fully describing the Cauchy stress by a term which is of the form of an additive hydrostatic stress $\varpi\mathbf{I}$. This has been physically interpreted as saying that for an incompressible material the constraint makes it such that one can add any hydrostatic stress onto any state of stress without altering its shape. The second equation states that for this material the entropy is fully determined from the free-energy, and is independent of p . The other relations, as stated above, require the free-energy to be independent of p and \mathbf{G} , and that $(\mathbf{q} \circ \mathbf{g})/\theta \leq 0$. It is common to set p equal to ϖ , removing all ambiguity in the role and definition of p . It should be noted that for incompressible materials the volume cannot change even with temperature. Therefore, this constraint does not even accommodate thermal expansion. One can construct an isothermally incompressible material that accommodates thermal expansion by making the volume change be fully determined by the temperature change, the value remaining constant if there is no temperature change. This will be considered in a later section.

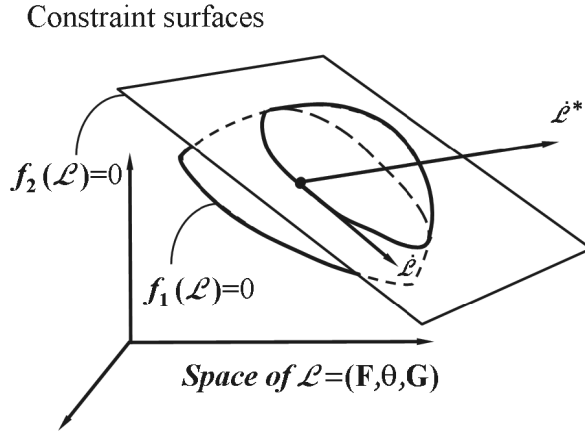


Figure 2. The schematic of the intersection of two constraint surfaces in the 13-dimensional loading space \mathcal{L} .

5. MULTIPLE MATERIAL CONSTRAINTS

In place of a single constraint, let there be n scalar constraints, each given by a constraint equation of the form

$$f_i(\mathbf{X}, \mathcal{L}) = 0, \quad (48)$$

where i is an integer between 1 and n . For each constraint an additional scalar variable p_i will be added to the constitutive equations so that

$$\psi = \psi^\dagger(\mathbf{X}, \mathcal{L}, p_1, \dots, p_n), \quad (49)$$

$$\mathbf{T} = \mathbf{T}^\dagger(\mathbf{X}, \mathcal{L}, p_1, \dots, p_n), \quad (50)$$

$$\eta = \eta^\dagger(\mathbf{X}, \mathcal{L}, p_1, \dots, p_n), \quad (51)$$

$$\mathbf{q} = \mathbf{q}^\dagger(\mathbf{X}, \mathcal{L}, p_1, \dots, p_n). \quad (52)$$

The differential form of the constraint equations can be written by setting each $\dot{f}_i = 0$ to obtain the equations

$$\partial_{\mathcal{L}}(f_i) \circ \dot{\mathcal{L}} = 0. \quad (53)$$

As in the case of a single constraint, one can introduce an arbitrary loading rate $\dot{\mathcal{L}}^*$ which can be used to extract an admissible loading rate $\dot{\mathcal{L}}$ that is compatible with all constraints and is written as

$$\dot{\mathcal{L}} = \dot{\mathcal{L}}^* - \alpha_j \partial_{\mathcal{L}}(f_j). \quad (54)$$

The n scalar coefficients α_i are selected such that $\dot{\mathcal{L}}$ is forced to satisfy all the constraints simultaneously. Substitution of this into the n constraints $\partial_{\mathcal{L}}(f_i) \circ \dot{\mathcal{L}} = 0$ results in the n equations

$$\partial_{\mathcal{L}}(f_i) \circ [\dot{\mathcal{L}}^* - \alpha_j \partial_{\mathcal{L}}(f_j)] = 0. \quad (55)$$

Reorganizing this results in the n equations

$$\partial_{\mathcal{L}}(f_i) \circ \partial_{\mathcal{L}}(f_j) \alpha_j = \partial_{\mathcal{L}}(f_i) \circ \dot{\mathcal{L}}^*. \quad (56)$$

Letting $A_{ij} = \partial_{\mathcal{L}}(f_i) \circ \partial_{\mathcal{L}}(f_j)$ and $c_i = \partial_{\mathcal{L}}(f_i) \circ \dot{\mathcal{L}}^*$, one gets a linear system of n equations described by the equation

$$A_{ij} \alpha_j = c_i, \quad (57)$$

where A_{ij} are the components of a real symmetric matrix of coefficients $[A]$. If $[A]$ is invertible, one can write

$$\alpha_i = A_{ij}^{-1} c_j. \quad (58)$$

The invertibility of the coefficient matrix $[A]$ is the key to the existence of α_i . If $[A]$ is not invertible, then there can only be non-trivial solutions if the loading is such that all the c_j are zero. In general, there is always the possibility that the constraints are inconsistent, or that the constraints “lock” all or part of the loading \mathcal{L} so that loading rate $\dot{\mathcal{L}}$ or part of it is forced to be zero. If the constraint conditions are totally incompatible, a loading \mathcal{L} cannot be found to satisfy all the constraints simultaneously, and, therefore, further consideration of this set of constraints is impractical. Yet, the locking of part or all of the response might still result in a material that is of practical interest. Figure 3 shows schematics of some of these possible cases.

It will be assumed that the constraints are consistent and $[A]$ is invertible so that one can always calculate the α_i . Under these conditions one obtains

$$\dot{\mathcal{L}} = \dot{\mathcal{L}}^* - \alpha_i \partial_{\mathcal{L}}(f_i) = \dot{\mathcal{L}}^* - A_{ij}^{-1} c_j \partial_{\mathcal{L}}(f_i) = \dot{\mathcal{L}}^* - A_{ij}^{-1} [\partial_{\mathcal{L}}(f_j) \circ \dot{\mathcal{L}}^*] \partial_{\mathcal{L}}(f_i). \quad (59)$$

The specific rates for \mathbf{F} , θ , and \mathbf{G} can be written as

$$\begin{aligned} \dot{\mathbf{F}} &= \dot{\mathbf{F}}^* - A_{ij}^{-1} [\partial_{\mathcal{L}}(f_j) \circ \dot{\mathcal{L}}^*] \partial_{\mathbf{F}}(f_i), \\ \dot{\theta} &= \dot{\theta}^* - A_{ij}^{-1} [\partial_{\mathcal{L}}(f_j) \circ \dot{\mathcal{L}}^*] \partial_{\theta}(f_i), \\ \dot{\mathbf{G}} &= \dot{\mathbf{G}}^* - A_{ij}^{-1} [\partial_{\mathcal{L}}(f_j) \circ \dot{\mathcal{L}}^*] \partial_{\mathbf{G}}(f_i). \end{aligned} \quad (60)$$

The material derivative of the free-energy under this condition is given by

$$\dot{\psi} = \partial_{\mathcal{L}}(\psi) \circ \dot{\mathcal{L}} + \partial_{p_i}(\psi) \dot{p}_i, \quad (61)$$

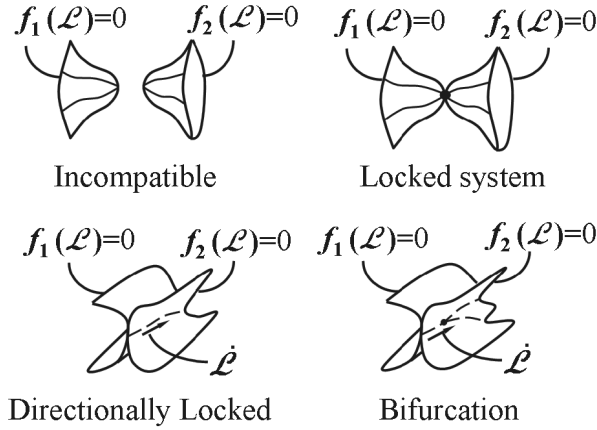


Figure 3. The schematic of some of the possible things that can happen when dealing with more than one constraint.

which can be reorganized as

$$\dot{\psi} = \{ \partial_{\mathcal{L}}(\psi) - [\partial_{\mathcal{L}}(\psi) \circ \partial_{\mathcal{L}}(f_i)] A_{ij}^{-1} \partial_{\mathcal{L}}(f_j) \} \circ \dot{\mathcal{L}}^* + \partial_{p_i}(\psi) \dot{p}_i. \quad (62)$$

Substitution of this and the above relations for $\dot{\mathbf{F}}$ and $\dot{\theta}$ into the Clausius–Duhem inequality will result in the inequality

$$\begin{aligned} & \rho \{ \partial_{\mathcal{L}}(\psi) - [\partial_{\mathcal{L}}(\psi) \circ \partial_{\mathcal{L}}(f_i)] A_{ij}^{-1} \partial_{\mathcal{L}}(f_j) \} \circ \dot{\mathcal{L}}^* + \rho \partial_{p_i}(\psi) \dot{p}_i \\ & - (\mathbf{T}^T \mathbf{F}^{-T}) : \{ \dot{\mathbf{F}}^* - A_{ij}^{-1} [\partial_{\mathcal{L}}(f_j) \circ \dot{\mathcal{L}}^*] \partial_{\mathbf{F}}(f_i) \} \\ & + \rho \eta \left\{ \dot{\theta}^* - A_{ij}^{-1} [\partial_{\mathcal{L}}(f_j) \circ \dot{\mathcal{L}}^*] \partial_{\theta}(f_i) \right\} + \frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0. \end{aligned} \quad (63)$$

As can be seen, this equation is linear in $\dot{\mathcal{L}}^*$ and \dot{p}_i so that one can organize it in the form of $14 + n$ terms where the first $13 + n$ terms each only contain one of the $13 + n$ components of $\dot{\mathcal{L}}^*$ and \dot{p}_i and the last term is $(\mathbf{q} \circ \mathbf{g})/\theta$. The multiplying factor for each of the $13 + n$ rates is independent of rate, so that the only way for the equation to be true for all $\dot{\mathcal{L}}^*$ and \dot{p}_i is that these multiplying factors each be equal to zero. The result of this process is the following equations:

$$\rho \partial_{\mathbf{F}}(\psi) + \varpi_j \partial_{\mathbf{F}}(f_j) - \mathbf{T}^T \mathbf{F}^{-T} = 0, \quad (64)$$

$$\rho \partial_{\theta}(\psi) + \varpi_j \partial_{\theta}(f_j) + \rho \eta = 0, \quad (65)$$

$$\rho \partial_{\mathbf{G}}(\psi) + \varpi_j \partial_{\mathbf{G}}(f_j) = 0, \quad (66)$$

$$\partial_{p_i}(\psi) = 0, \quad (67)$$

$$\frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0, \quad (68)$$

where

$$\varpi_j = - \left\{ \rho \partial_{\mathcal{L}}(\psi) \circ \partial_{\mathcal{L}}(f_i) - (\mathbf{T}^T \mathbf{F}^{-T}) : \partial_{\mathbf{F}}(f_i) + \rho \eta \partial_{\theta}(f_i) \right\} A_{ij}^{-1} \quad (69)$$

Reorganization of the equations now yields

$$\mathbf{T}^T = \mathbf{T}_E^T + \varpi_j \partial_{\mathbf{F}}(f_j) \mathbf{F}^T, \quad (70)$$

$$\eta = \eta_E - \frac{\varpi_j}{\rho} \partial_{\theta}(f_j), \quad (71)$$

$$\partial_{\mathbf{G}}(\psi) = -\frac{\varpi_j}{\rho} \partial_{\mathbf{G}}(f_j), \quad (72)$$

$$\partial_{p_i}(\psi) = 0, \quad (73)$$

$$\frac{1}{\theta} \mathbf{q} \circ \mathbf{g} \leq 0. \quad (74)$$

As in the case of a single constraint, the third and fourth equations can be taken as a constraint on the form of the free-energy. In particular, *free-energy can never depend on any of the p_i* . Also, if the constraint conditions do not depend on the temperature gradient \mathbf{G} , then the free-energy cannot depend on temperature gradient. The Cauchy stress and entropy partially inherit such characteristics from the free-energy through the first two equations, with the explicit dependence on p_i given only in the ϖ_i . The last equation restricts the possible direction of heat flux in relation to the direction of the temperature gradient. These equations must still be augmented by the n constraint equations $f_i(\mathbf{X}, \mathbf{F}, \theta, \mathbf{G}) = 0$.

6. GROUPING OF TERMS

It is common in purely mechanical theories with one internal constraint to decompose the Cauchy stress into a *reaction stress*, \mathbf{T}_R , due to the constraint, and an *extra stress*, \mathbf{T}_E , that is prescribed constitutively in terms of a function of (\mathbf{X}, \mathbf{F}) (see [7, 9]).⁴ This additive decomposition is written as

$$\mathbf{T} = \mathbf{T}_E + \mathbf{T}_R. \quad (75)$$

The decomposition is not unique, but normally refers to the selection

$$\mathbf{T}_E^T = \rho \partial_{\mathbf{F}}(\psi) \mathbf{F}^T, \quad \mathbf{T}_R^T = \varpi \partial_{\mathbf{F}}(f) \mathbf{F}^T. \quad (76)$$

This form of the decomposition need not necessarily result in a physically meaningful separation of terms. One method of obtaining more physically meaningful terms is to remove from \mathbf{T}_E the portion along \mathbf{T}_R . To do this one can write

$$\mathbf{T} = (\mathbf{T}_E - a\mathbf{T}_R) + (1 + a)\mathbf{T}_R, \quad (77)$$

with the condition that $\mathbf{T} : \mathbf{T}_R = (1 + a)\mathbf{T}_R : \mathbf{T}_R$. This requires that

$$a = \frac{\mathbf{T}_E : \mathbf{T}_R}{\mathbf{T}_R : \mathbf{T}_R}. \quad (78)$$

As can be seen by substitution, the term $\mathbf{T}_E^* = \mathbf{T}_E - a\mathbf{T}_R$, that is the newly defined extra stress, becomes

$$\mathbf{T}_E^{*T} = \mathbf{T}_E^T - a\mathbf{T}_R^T = \rho \partial_{\mathbf{F}}(\psi) \mathbf{F}^T - \frac{[\rho \partial_{\mathbf{F}}(\psi) \mathbf{F}^T] : [\partial_{\mathbf{F}}(f) \mathbf{F}^T]}{[\partial_{\mathbf{F}}(f) \mathbf{F}^T] : [\partial_{\mathbf{F}}(f) \mathbf{F}^T]} \partial_{\mathbf{F}}(f) \mathbf{F}^T, \quad (79)$$

and is independent of the indeterminate function p . One can take $\varpi^* = (1 + a)\varpi$ and define a new reaction stress $\mathbf{T}_R^{*T} = \varpi^* \partial_{\mathbf{F}}(f) \mathbf{F}^T$ to obtain

$$\mathbf{T} = \mathbf{T}_E^* + \mathbf{T}_R^*, \quad (80)$$

for which the extra stress \mathbf{T}_E^* has no projection along the reaction stress \mathbf{T}_R^* , so that $\mathbf{T}_E^* : \mathbf{T}_R^* = 0$. For example, for the case of the incompressible material one obtains

$$\mathbf{T}_E^* = \mathbf{T}_E - \frac{1}{3} \text{tr}(\mathbf{T}_E) \mathbf{I}, \quad \mathbf{T}_R^* = \frac{1}{3} \text{tr}(\mathbf{T}) \mathbf{I}. \quad (81)$$

As can be seen, for this constraint the extra stress \mathbf{T}_E^* is traceless, so that ϖ^* becomes equal to the average normal Cauchy stress. One may interpret this as stating that one can prescribe a constitutive equation for all but the trace of the Cauchy stress. Therefore, the indeterminacy introduced by this constraint is in the value of the average normal Cauchy stress $T_{\text{ave}} = \text{tr}(\mathbf{T})/3$.

Even though the method described above works very well for purely mechanical theories, it is not the only method for grouping terms together. There are many methods for separating the different terms. Each method is distinguished by the criteria used to accomplish the separation. Here three different methods of decoupling the terms will be studied. This is done in the context of multiple constraints, the single constraint being a special case. In each method one arrives at an expression for Cauchy stress and entropy of the form

$$\mathbf{T} = \mathbf{T}_E^{(j)} + \mathbf{T}_R^{(j)}, \quad \eta = \eta_E^{(j)} + \eta_R^{(j)}, \quad (82)$$

where j refers to the method used. In each case $\varpi_i = \bar{\varpi}_i^{(j)} + \varpi_i^{*(j)}$ and

$$\mathbf{T}_E^{(j)T} = \mathbf{T}_E^T + \bar{\varpi}_i^{(j)} \partial_{\mathbf{F}}(f_i) \mathbf{F}^T, \quad \mathbf{T}_R^{(j)T} = \varpi_i^{*(j)} \partial_{\mathbf{F}}(f_i) \mathbf{F}^T, \quad (83)$$

$$\eta_E^{(j)} = \eta_E - \frac{\bar{\varpi}_i^{(j)}}{\rho} \partial_{\theta}(f_i), \quad \eta_R^{(j)} = -\frac{\varpi_i^{*(j)}}{\rho} \partial_{\theta}(f_i). \quad (84)$$

Each method is distinguished by the criteria used to separate ϖ_i into $\varpi_i = \bar{\varpi}_i^{(j)} + \varpi_i^{*(j)}$.

Method 1.⁵ This method removes from $\rho\partial_{\mathcal{L}}(\psi)$ its projections along each $\partial_{\mathcal{L}}(f_i)$ so that the remainder is orthogonal to all $\partial_{\mathcal{L}}(f_i)$. In this method one starts from the equations

$$\begin{aligned}\mathbf{T}^T\mathbf{F}^{-T} &= \rho\partial_{\mathbf{F}}(\psi) + \varpi_i\partial_{\mathbf{F}}(f_i), \\ -\rho\eta &= \rho\partial_{\theta}(\psi) + \varpi_i\partial_{\theta}(f_i), \\ 0 &= \rho\partial_{\mathbf{G}}(\psi) + \varpi_i\partial_{\mathbf{G}}(f_i),\end{aligned}\quad (85)$$

and separate each ϖ_i into two terms $\varpi_i = \bar{\varpi}_i^{(1)} + \varpi_i^{*(1)}$ with the condition that

$$[\rho\partial_{\mathcal{L}}(\psi) + \bar{\varpi}_i^{(1)}\partial_{\mathcal{L}}(f_i)] \circ \partial_{\mathcal{L}}(f_j) = 0, \quad (86)$$

for every j between 1 and n . This condition forces $\rho\partial_{\mathcal{L}}(\psi) + \bar{\varpi}_i^{(1)}\partial_{\mathcal{L}}(f_i)$ to be orthogonal to every $\partial_{\mathcal{L}}(f_j)$. This system of equations can be solved for the unknown $\bar{\varpi}_i^{(1)}$ to obtain

$$\bar{\varpi}_i^{(1)} = -\rho\partial_{\mathcal{L}}(\psi) \circ \partial_{\mathcal{L}}(f_j) A_{ji}^{-1}. \quad (87)$$

Since ρ , ψ , f_i , and A_{ji}^{-1} are all independent of the indeterminate functions p_i , each $\bar{\varpi}_i^{(1)}$ is independent of p_i and so is the term $\rho\partial_{\mathcal{L}}(\psi) + \bar{\varpi}_i^{(1)}\partial_{\mathcal{L}}(f_i)$. The determination of $\varpi_i^{*(1)}$ can be done through the relation

$$\rho\partial_{\mathcal{L}}(\psi) \circ \partial_{\mathcal{L}}(f_j) = \rho\partial_{\mathbf{F}}(\psi) : \partial_{\mathbf{F}}(f_j) + \rho\partial_{\theta}(\psi)\partial_{\theta}(f_j) + \rho\partial_{\mathbf{G}}(\psi) \circ \partial_{\mathbf{G}}(f_j). \quad (88)$$

After substitution of (85) and elimination of terms this yields

$$\varpi_i^{*(1)} = [(\mathbf{T}^T\mathbf{F}^{-T}) : \partial_{\mathbf{F}}(f_j) - \rho\eta\partial_{\theta}(f_j)] A_{ji}^{-1}. \quad (89)$$

As can be seen, each $\varpi_i^{*(1)}$ contains both the Cauchy stress and entropy, both of which can depend on the indeterminate p_i . Therefore, the indeterminacy in the prescription of the response in terms of the loading is concentrated in each $\varpi_i^{*(1)}$. It is to be noted that the calculation of $\bar{\varpi}_i^{(1)}$ and $\varpi_i^{*(1)}$ depends on the invertibility of $[A]$, which has already been assumed in obtaining (85).

Method 2. This method is based on making $\mathbf{T}_E^{(2)T}\mathbf{F}^{-T}$ simultaneously orthogonal to all $\partial_{\mathbf{F}}(f_i)$. This is done by starting with the equation

$$\mathbf{T}^T\mathbf{F}^{-T} = \rho\partial_{\mathbf{F}}(\psi) + \varpi_i\partial_{\mathbf{F}}(f_i) \quad (90)$$

and separation of each ϖ_i into two terms $\varpi_i = \bar{\varpi}_i^{(2)} + \varpi_i^{*(2)}$ with the condition that

$$[\rho\partial_{\mathbf{F}}(\psi) + \bar{\varpi}_i^{(2)}\partial_{\mathbf{F}}(f_i)] : \partial_{\mathbf{F}}(f_j) = 0, \quad (91)$$

for every j . This condition forces $\rho \partial_{\mathbf{F}}(\psi) + \bar{\varpi}_i \partial_{\mathbf{F}}(f_i)$ to be orthogonal to every $\partial_{\mathbf{F}}(f_j)$. This system of equations can be solved for the unknown $\bar{\varpi}_i^{(2)}$ to obtain

$$\bar{\varpi}_i^{(2)} = -(\mathbf{T}_E^T \mathbf{F}^{-T}) : \partial_{\mathbf{F}}(f_j) K_{ji}^{-1} \quad (92)$$

if the inverse of the matrix $[K]$ with components $K_{ij} = \partial_{\mathbf{F}}(f_i) : \partial_{\mathbf{F}}(f_j)$ exists. Since ρ , ψ , f_i , and K_{ji}^{-1} are all independent of the indeterminate functions p_i , each $\bar{\varpi}_i^{(2)}$ is independent of p_i and so is the term $\rho \partial_{\mathbf{F}}(\psi) + \bar{\varpi}_i^{(2)} \partial_{\mathbf{F}}(f_i)$. The determination of $\varpi_i^{*(2)}$ can be done through substitution back into the first equation to obtain

$$\varpi_i^{*(2)} = (\mathbf{T}^T \mathbf{F}^{-T}) : \partial_{\mathbf{F}}(f_j) K_{ji}^{-1}. \quad (93)$$

As can be seen, each $\varpi_i^{*(2)}$ contains the Cauchy stress, which can depend on the indeterminate p_i . Unlike the first method, this method is more suited for mechanical constraints. The method can still be used in other cases when some of the constraints are non-mechanical. This is done by only solving for the $\bar{\varpi}_i^{(2)}$ and $\varpi_i^{*(2)}$ associated with the mechanical constraints.

Method 3. This method is based on making $\mathbf{T}_E^{(3)T}$ simultaneously orthogonal to all $\partial_{\mathbf{F}}(f_i) \mathbf{F}^T$. This method, as was the case for Method 2, is best suited for mechanical constraints. The orthogonalization is done by starting from

$$\mathbf{T}^T = \rho \partial_{\mathbf{F}}(\psi) \mathbf{F}^T + \varpi_i \partial_{\mathbf{F}}(f_i) \mathbf{F}^T \quad (94)$$

and separation of each ϖ_i into two terms $\varpi_i = \bar{\varpi}_i^{(3)} + \varpi_i^{*(3)}$ with the condition that

$$\left[\rho \partial_{\mathbf{F}}(\psi) \mathbf{F}^T + \bar{\varpi}_i^{(3)} \partial_{\mathbf{F}}(f_i) \mathbf{F}^T \right] : [\partial_{\mathbf{F}}(f_j) \mathbf{F}^T] = 0, \quad (95)$$

for every j . This system of equations can be solved for the unknown $\bar{\varpi}_i$ to obtain

$$\bar{\varpi}_i^{(3)} = -\mathbf{T}_E^T : [\partial_{\mathbf{F}}(f_j) \mathbf{F}^T] K_{ji}^{-1} \quad (96)$$

if the inverse of the matrix $[K]$ with components $K_{ij} = [\partial_{\mathbf{F}}(f_i) \mathbf{F}^T] : [\partial_{\mathbf{F}}(f_j) \mathbf{F}^T]$ exists. Each $\bar{\varpi}_i^{(3)}$ becomes independent of p_i and so is the term $\rho \partial_{\mathbf{F}}(\psi) + \bar{\varpi}_i^{(3)} \partial_{\mathbf{F}}(f_i)$. The determination of $\varpi_i^{*(3)}$ can be done through substitution back into the first equation to obtain

$$\varpi_i^{*(3)} = \mathbf{T}^T : [\partial_{\mathbf{F}}(f_j) \mathbf{F}^T] K_{ji}^{-1}. \quad (97)$$

As can be seen, each $\varpi_i^{*(3)}$ contains the Cauchy stress, which can depend on the indeterminate p_i . As for the second method, this method is best suited for mechanical constraints. The method can still be used when some of the constraints are non-mechanical. This is done by only solving for the $\bar{\varpi}_i^{(3)}$ and $\varpi_i^{*(3)}$ associated with the mechanical constraints.

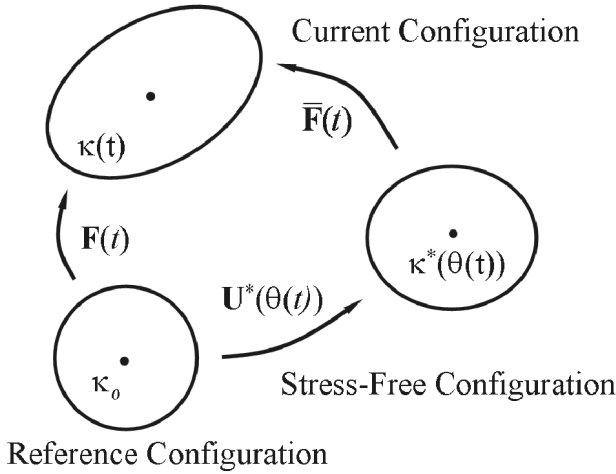


Figure 4. Reference, current, and the current stress-free configurations of the neighborhood of influence of a point. The current stress-free configuration is obtained by isothermally unloading the neighborhood of the point and then rigidly rotating until the deformation gradient \mathbf{U}^* describing the deformation from κ_0 to κ^* is symmetric.

7. EXAMPLES OF SINGLE CONSTRAINTS

The following are a number of examples of single constraints. Notable among them is the last example imposing a zero temperature gradient normal to a material surface, which results in an indeterminate shear stress on the surface in the direction of the temperature gradient, among other results. Also, among the examples methods are introduced to include isotropic and anisotropic thermal expansion.

Isothermally incompressible: A material that has constant volume at constant temperature is called an isothermally incompressible material. Consider the reference, current, and current stress-free configurations of the neighborhood of a material point as shown in Figure 4, where the current stress-free configuration is obtained by isothermally unloading the neighborhood of influence of each material point and then rigidly rotating it so that the deformation gradient \mathbf{U}^* becomes symmetric. The deformation gradient \mathbf{U}^* describes the thermal expansion at the material point, and for the case of isotropic thermal expansion takes the form $\mathbf{U}^* = J^*(\theta)^{1/3}\mathbf{I}$ if the reference configuration coincides with a real unloaded configuration of the material neighborhood. In general, for anisotropic thermal expansion, \mathbf{U}^* is a general symmetric tensor function of the current temperature. The constraint associated with an isothermally incompressible material can be written as $\det(\bar{\mathbf{F}}) = 1$, and in view of the relation $\mathbf{F} = \bar{\mathbf{F}}\mathbf{U}^*$, this can be written as

$$\det(\mathbf{F}) = J^*(\theta), \quad (98)$$

where $J^*(\theta) = \det(\mathbf{U}^*)$ is a function of temperature giving the volume ratio associated with the unloaded neighborhood at each temperature. An example of this function would be $J^*(\theta) = J_o + \beta(\theta - \theta_o)$, where J_o is the volume ratio at a reference temperature θ_o , and β is the volumetric coefficient of thermal expansion. The constraint function in this case is $f = \det(\mathbf{F}) - J^*(\theta) = 0$ with the partial derivatives

$$\partial_{\mathbf{F}}(f) = \det(\mathbf{F})\mathbf{F}^{-T}, \quad \partial_{\theta}(f) = -\frac{dJ^*(\theta)}{d\theta}. \quad (99)$$

These partial derivatives result in the following expressions for Cauchy stress and entropy:

$$\mathbf{T} = \mathbf{T}_E + \varpi \det(\mathbf{F})\mathbf{I}, \quad \eta = \eta_E + \frac{\varpi}{\rho} \frac{dJ^*}{d\theta}. \quad (100)$$

Since the constraint does not depend on \mathbf{G} , it is also concluded that specific free-energy is independent of \mathbf{G} . Without any loss of generality, one can define the indeterminate function $p = \varpi \det(\mathbf{F})$ and replace this into the expressions to obtain

$$\mathbf{T} = \mathbf{T}_E + p\mathbf{I}, \quad \eta = \eta_E + \frac{p}{\rho_o} \frac{dJ^*}{d\theta}, \quad (101)$$

where ρ_o is the density at the reference temperature θ_o , and $\rho \det(\mathbf{F}) = \rho_o$ follows from conservation of mass. The grouping of terms in the three methods presented result in the following expressions:

$$\begin{aligned} \mathbf{T}_E^{(1)T} &= \mathbf{T}_E^T - \frac{\text{tr}(\mathbf{T}_E^T \mathbf{B}^{-1}) - \rho \eta_E \frac{1}{J^*} \frac{dJ^*}{d\theta}}{\text{tr}(\mathbf{B}^{-1}) + \frac{1}{J^{*2}} \left(\frac{dJ^*}{d\theta} \right)^2} \mathbf{I}, \\ \mathbf{T}_R^{(1)T} &= \frac{\text{tr}(\mathbf{T}^T \mathbf{B}^{-1}) - \rho \eta \frac{1}{J^*} \frac{dJ^*}{d\theta}}{\text{tr}(\mathbf{B}^{-1}) + \frac{1}{J^{*2}} \left(\frac{dJ^*}{d\theta} \right)^2} \mathbf{I}, \end{aligned} \quad (102)$$

$$\begin{aligned} \eta_E^{(1)} &= \eta_E - \frac{\text{tr}(\mathbf{T}_E^T \mathbf{B}^{-1}) - \rho \eta_E \frac{1}{J^*} \frac{dJ^*}{d\theta}}{\rho \left[\text{tr}(\mathbf{B}^{-1}) + \frac{1}{J^{*2}} \left(\frac{dJ^*}{d\theta} \right)^2 \right]} \frac{dJ^*}{d\theta}, \\ \eta_R^{(1)} &= \frac{\text{tr}(\mathbf{T}^T \mathbf{B}^{-1}) - \rho \eta \frac{1}{J^*} \frac{dJ^*}{d\theta}}{\rho \left[\text{tr}(\mathbf{B}^{-1}) + \frac{1}{J^{*2}} \left(\frac{dJ^*}{d\theta} \right)^2 \right]} \frac{dJ^*}{d\theta}, \end{aligned} \quad (103)$$

$$\mathbf{T}_E^{(2)T} = \mathbf{T}_E^T - \frac{\text{tr}(\mathbf{T}_E^T \mathbf{B}^{-1})}{\text{tr}(\mathbf{B}^{-1})} \mathbf{I}, \quad \mathbf{T}_R^{(2)T} = \frac{\text{tr}(\mathbf{T}^T \mathbf{B}^{-1})}{\text{tr}(\mathbf{B}^{-1})} \mathbf{I}, \quad (104)$$

$$\eta_E^{(2)} = \eta_E - \frac{\text{tr}(\mathbf{T}_E^T \mathbf{B}^{-1})}{\rho \text{tr}(\mathbf{B}^{-1})} \frac{dJ^*}{d\theta}, \quad \eta_R^{(2)} = \frac{\text{tr}(\mathbf{T}^T \mathbf{B}^{-1})}{\rho \text{tr}(\mathbf{B}^{-1})} \frac{dJ^*}{d\theta}, \quad (105)$$

$$\mathbf{T}_E^{(3)T} = \mathbf{T}_E^T - \frac{\text{tr}(\mathbf{T}_E)}{3} \mathbf{I}, \quad \mathbf{T}_R^{(3)T} = \frac{\text{tr}(\mathbf{T})}{3} \mathbf{I}, \quad (106)$$

$$\eta_E^{(3)} = \eta_E - \frac{\text{tr}(\mathbf{T}_E)}{3\rho} \frac{dJ^*}{d\theta}, \quad \eta_R^{(3)} = \frac{\text{tr}(\mathbf{T})}{3\rho} \frac{dJ^*}{d\theta}. \quad (107)$$

Note that $J \text{tr}(\mathbf{T}^T \mathbf{B}^{-1}) = \text{tr}(\mathbf{P})$, where \mathbf{P} is the second Piola–Kirchhoff stress tensor. The third method results in the most physically meaningful separation since $T_{\text{ave}} = \text{tr}(\mathbf{T})/3$ is the average normal stress.

Isothermally inextensible: A material that at constant temperature is inextensible along a given material direction is called an isothermally inextensible material. Consider a material in which the material line element along the direction given by the unit vector $\hat{\mathbf{h}}_o$ in the reference configuration at temperature θ_o has a length dl_o and the same line element in the current configuration at temperature θ has the length $dl = \lambda(\theta)dl_o$ and is along the direction given by the unit vector $\hat{\mathbf{h}} = \mathbf{F}\hat{\mathbf{h}}_o/\lambda$. Using the relation $dl^2 = \hat{\mathbf{h}}_o \circ (\mathbf{C}\hat{\mathbf{h}}_o)dl_o^2$, this constraint can be written as

$$\hat{\mathbf{h}}_o \circ (\mathbf{C}\hat{\mathbf{h}}_o) = \lambda^2(\theta). \quad (108)$$

Using the constraint function $f = \hat{\mathbf{h}}_o \circ (\mathbf{C}\hat{\mathbf{h}}_o) - \lambda^2(\theta) = 0$, the partial derivatives are

$$\partial_{\mathbf{F}}(f) = 2\lambda\hat{\mathbf{h}} \otimes \hat{\mathbf{h}}_o, \quad \partial_{\theta}(f) = -2\lambda \frac{d\lambda}{d\theta}. \quad (109)$$

These result in the following expressions for Cauchy stress and entropy:

$$\mathbf{T} = \mathbf{T}_E + 2\varpi\lambda^2 \hat{\mathbf{h}} \otimes \hat{\mathbf{h}}, \quad \eta = \eta_E + \frac{2\varpi}{\rho} \lambda \frac{d\lambda}{d\theta}. \quad (110)$$

The grouping of terms in the three methods presented result in the following expressions:

$$\begin{aligned} \mathbf{T}_E^{(1)T} &= \mathbf{T}_E^T - \frac{\lambda^2 \hat{\mathbf{h}} \circ (\mathbf{T}_E^T \mathbf{B}^{-1} \hat{\mathbf{h}}) + 2\rho\eta_E \frac{1}{\lambda} \frac{d\lambda}{d\theta}}{1 + \left(\frac{d\lambda}{d\theta}\right)^2} \hat{\mathbf{h}} \otimes \hat{\mathbf{h}}, \\ \mathbf{T}_R^{(1)T} &= \frac{\lambda^2 \hat{\mathbf{h}} \circ (\mathbf{T}^T \mathbf{B}^{-1} \hat{\mathbf{h}}) + 2\rho\eta \frac{1}{\lambda} \frac{d\lambda}{d\theta}}{1 + \left(\frac{d\lambda}{d\theta}\right)^2} \hat{\mathbf{h}} \otimes \hat{\mathbf{h}}, \end{aligned} \quad (111)$$

$$\begin{aligned}\eta_E^{(1)} &= \eta_E - \frac{\lambda}{2\rho} \frac{\hat{\mathbf{h}} \circ (\mathbf{T}_E^T \mathbf{B}^{-1} \hat{\mathbf{h}}) + 2\rho \eta_E \frac{1}{\lambda} \frac{d\lambda}{d\theta}}{1 + \left(\frac{d\lambda}{d\theta}\right)^2} \frac{d\lambda}{d\theta}, \\ \eta_R^{(1)} &= \frac{\lambda}{2\rho} \frac{\hat{\mathbf{h}} \circ (\mathbf{T}^T \mathbf{B}^{-1} \hat{\mathbf{h}}) + 2\rho \eta \frac{1}{\lambda} \frac{d\lambda}{d\theta}}{1 + \left(\frac{d\lambda}{d\theta}\right)^2} \frac{d\lambda}{d\theta},\end{aligned}\quad (112)$$

$$\mathbf{T}_E^{(2)T} = \mathbf{T}_E^T - \frac{\lambda^2}{2} \hat{\mathbf{h}} \circ (\mathbf{T}_E^T \mathbf{B}^{-1} \hat{\mathbf{h}}) \hat{\mathbf{h}} \otimes \hat{\mathbf{h}}, \quad \mathbf{T}_R^{(2)T} = \frac{\lambda^2}{2} \hat{\mathbf{h}} \circ (\mathbf{T}^T \mathbf{B}^{-1} \hat{\mathbf{h}}) \hat{\mathbf{h}} \otimes \hat{\mathbf{h}}, \quad (113)$$

$$\eta_E^{(2)} = \eta_E - \frac{\lambda}{2\rho} \hat{\mathbf{h}} \circ (\mathbf{T}_E^T \mathbf{B}^{-1} \hat{\mathbf{h}}) \frac{d\lambda}{d\theta}, \quad \eta_R^{(2)} = \frac{\lambda}{2\rho} \hat{\mathbf{h}} \circ (\mathbf{T}^T \mathbf{B}^{-1} \hat{\mathbf{h}}) \frac{d\lambda}{d\theta}, \quad (114)$$

$$\mathbf{T}_E^{(3)T} = \mathbf{T}_E^T - \lambda^2 \hat{\mathbf{h}} \circ (\mathbf{T}_E^T \hat{\mathbf{h}}) \hat{\mathbf{h}} \otimes \hat{\mathbf{h}}, \quad \mathbf{T}_R^{(3)T} = \lambda^2 \hat{\mathbf{h}} \circ (\mathbf{T}^T \hat{\mathbf{h}}) \hat{\mathbf{h}} \otimes \hat{\mathbf{h}}, \quad (115)$$

$$\eta_E^{(3)} = \eta_E - \frac{\lambda^2 \hat{\mathbf{h}} \circ (\mathbf{T}_E^T \hat{\mathbf{h}})}{\rho} \frac{d\lambda}{d\theta}, \quad \eta_R^{(3)} = \frac{\lambda^2 \hat{\mathbf{h}} \circ (\mathbf{T}^T \hat{\mathbf{h}})}{\rho} \frac{d\lambda}{d\theta}. \quad (116)$$

Obviously the third method results in the most physically meaningful separation since $N = \hat{\mathbf{h}} \circ (\mathbf{T}^T \hat{\mathbf{h}})$ is the normal load on the surface perpendicular to the direction of inextensibility, and, therefore, is the normal traction that is made indeterminate by the constraint.

Isothermally Bell constrained with anisotropic thermal expansion: Here an isothermally Bell constrained material will be considered for which $\text{tr}(\bar{\mathbf{V}}) = 3$, where $\bar{\mathbf{V}}$ is the left symmetric factor in the polar decomposition of $\bar{\mathbf{F}}$ shown in Figure 4. The only difference with the last example is the fact that the unloaded thermal expansion may be represented by a symmetric \mathbf{U}^* other than equal triaxial extension. Denoting by $\bar{\mathbf{R}}$ and $\bar{\mathbf{U}}$ the orthogonal factor and the right symmetric factor in the polar decomposition of $\bar{\mathbf{F}}$, it can be shown that for $f = \text{tr}(\bar{\mathbf{V}}) - 3$ on has

$$\dot{f} = (\bar{\mathbf{R}}\mathbf{U}^{*-1}) : \dot{\bar{\mathbf{F}}} - (\mathbf{U}^{*-1}\bar{\mathbf{U}}) : \frac{d\mathbf{U}^*}{d\theta} \dot{\theta}, \quad (117)$$

so that one will obtain the partial derivatives

$$\partial_{\bar{\mathbf{F}}}(f) = \bar{\mathbf{R}}\mathbf{U}^{*-1}, \quad \partial_{\theta}(f) = -(\mathbf{U}^{*-1}\bar{\mathbf{U}}) : \frac{d\mathbf{U}^*}{d\theta}. \quad (118)$$

These result in the following expressions for Cauchy stress and entropy:

$$\mathbf{T} = \mathbf{T}_E + \varpi \bar{\mathbf{V}}, \quad \eta = \eta_E + \frac{\varpi}{\rho} (\mathbf{U}^{*-1}\bar{\mathbf{U}}) : \frac{d\mathbf{U}^*}{d\theta}. \quad (119)$$

The grouping of terms in the three methods presented result in the following expressions:

$$\mathbf{T}_E^{(1)T} = \mathbf{T}_E^T - \frac{\text{tr}(\mathbf{T}_E \bar{\mathbf{V}}^{-1}) + \rho \eta_E (\mathbf{U}^{*-1} \bar{\mathbf{U}}) : \frac{d\mathbf{U}^*}{d\theta}}{3 + \left[(\mathbf{U}^{*-1} \bar{\mathbf{U}}) : \frac{d\mathbf{U}^*}{d\theta} \right]^2} \bar{\mathbf{V}}, \quad (120)$$

$$\mathbf{T}_R^{(1)T} = \frac{\text{tr}(\mathbf{T} \bar{\mathbf{V}}^{-1}) + \rho \eta (\mathbf{U}^{*-1} \bar{\mathbf{U}}) : \frac{d\mathbf{U}^*}{d\theta}}{3 + \left[(\mathbf{U}^{*-1} \bar{\mathbf{U}}) : \frac{d\mathbf{U}^*}{d\theta} \right]^2} \bar{\mathbf{V}}, \quad (121)$$

$$\eta_E^{(1)} = \eta_E - \frac{\text{tr}(\mathbf{T}_E \bar{\mathbf{V}}^{-1}) + \rho \eta_E (\mathbf{U}^{*-1} \bar{\mathbf{U}}) : \frac{d\mathbf{U}^*}{d\theta}}{2\rho \left[3 + \left[(\mathbf{U}^{*-1} \bar{\mathbf{U}}) : \frac{d\mathbf{U}^*}{d\theta} \right]^2 \right]} (\mathbf{U}^{*-1} \bar{\mathbf{U}}) : \frac{d\mathbf{U}^*}{d\theta}, \quad (122)$$

$$\eta_R^{(1)} = \frac{\text{tr}(\mathbf{T} \bar{\mathbf{V}}^{-1}) + \rho \eta (\mathbf{U}^{*-1} \bar{\mathbf{U}}) : \frac{d\mathbf{U}^*}{d\theta}}{2\rho \left[3 + \left[(\mathbf{U}^{*-1} \bar{\mathbf{U}}) : \frac{d\mathbf{U}^*}{d\theta} \right]^2 \right]} (\mathbf{U}^{*-1} \bar{\mathbf{U}}) : \frac{d\mathbf{U}^*}{d\theta}, \quad (123)$$

$$\mathbf{T}_E^{(2)T} = \mathbf{T}_E^T - \frac{\text{tr}(\mathbf{T}_E \bar{\mathbf{V}}^{-1})}{3} \bar{\mathbf{V}}, \quad \mathbf{T}_R^{(2)T} = \frac{\text{tr}(\mathbf{T} \bar{\mathbf{V}}^{-1})}{3} \bar{\mathbf{V}}, \quad (124)$$

$$\eta_E^{(2)} = \eta_E - \frac{\text{tr}(\mathbf{T}_E \bar{\mathbf{V}}^{-1})}{3\rho} (\mathbf{U}^{*-1} \bar{\mathbf{U}}) : \frac{d\mathbf{U}^*}{d\theta}, \quad \eta_R^{(2)} = \frac{\text{tr}(\mathbf{T} \bar{\mathbf{V}}^{-1})}{3\rho} (\mathbf{U}^{*-1} \bar{\mathbf{U}}) : \frac{d\mathbf{U}^*}{d\theta}, \quad (125)$$

$$\mathbf{T}_E^{(3)T} = \mathbf{T}_E^T - \frac{\text{tr}(\mathbf{T}_E \bar{\mathbf{V}})}{\text{tr}(\mathbf{B})} \bar{\mathbf{V}}, \quad \mathbf{T}_R^{(3)T} = \frac{\text{tr}(\mathbf{T} \bar{\mathbf{V}})}{\text{tr}(\mathbf{B})} \bar{\mathbf{V}}, \quad (126)$$

$$\eta_E^{(3)} = \eta_E - \frac{\text{tr}(\mathbf{T}_E \bar{\mathbf{V}})}{\rho \text{tr}(\mathbf{B})} (\mathbf{U}^{*-1} \bar{\mathbf{U}}) : \frac{d\mathbf{U}^*}{d\theta}, \quad \eta_R^{(3)} = \frac{\text{tr}(\mathbf{T} \bar{\mathbf{V}})}{\rho \text{tr}(\mathbf{B})} (\mathbf{U}^{*-1} \bar{\mathbf{U}}) : \frac{d\mathbf{U}^*}{d\theta}. \quad (127)$$

General constraints of the form $\bar{f}(\bar{\mathbf{F}}, \theta) = 0$ with anisotropic thermal expansion: Under consideration here is a constraint that is naturally described by the deformation gradient $\bar{\mathbf{F}}$ which is the current deformation gradient evaluated relative to the stress-free configuration for the current temperature as described in Figure 4. The time rate of change of \bar{f} is given by

$$\dot{\bar{f}} = \partial_{\bar{\mathbf{F}}}(\bar{f}) : \dot{\bar{\mathbf{F}}} + \partial_{\theta}(\bar{f})\dot{\theta}. \quad (128)$$

In view of the relation $\mathbf{F} = \bar{\mathbf{F}}\mathbf{U}^*$, one has

$$\dot{\bar{\mathbf{F}}} = \dot{\mathbf{F}}\mathbf{U}^{*-1} + \mathbf{F} \frac{d\mathbf{U}^{*-1}}{dt}. \quad (129)$$

Combining this with the relation⁶

$$\frac{d\mathbf{U}^{*-1}}{dt} = -\mathbf{U}^{*-1}\dot{\mathbf{U}}^*\mathbf{U}^{*-1}, \quad (130)$$

and substituting back into \dot{f} results in

$$\dot{f} = [\partial_{\bar{\mathbf{F}}}(\bar{f})\mathbf{U}^{*-1}] : \dot{\mathbf{F}} + \left\{ \partial_{\theta}(\bar{f}) - [\bar{\mathbf{F}}^T \partial_{\bar{\mathbf{F}}}(\bar{f})\mathbf{U}^{*-1}] : \frac{d\mathbf{U}^*}{d\theta} \right\} \dot{\theta}. \quad (131)$$

It follows that if the constraint is written in the form $f(\mathbf{F}, \theta)$, as was done in the developments presented in the previous sections, the needed partial derivatives are

$$\partial_{\mathbf{F}}(f) = \partial_{\bar{\mathbf{F}}}(\bar{f})\mathbf{U}^{*-1}, \quad \partial_{\theta}(f) = \partial_{\theta}(\bar{f}) - [\bar{\mathbf{F}}^T \partial_{\bar{\mathbf{F}}}(\bar{f})\mathbf{U}^{*-1}] : \frac{d\mathbf{U}^*}{d\theta}, \quad \partial_{\mathbf{G}}(f) = 0. \quad (132)$$

Therefore, if the constraint is naturally stated in terms of deformations from the stress-free configuration at the current temperature, for general anisotropic thermal expansion the partial derivatives can be calculated very simply using these expressions. The expression for Cauchy stress and entropy will be

$$\mathbf{T}^T = \mathbf{T}_E^T + \varpi \partial_{\bar{\mathbf{F}}}(\bar{f})\bar{\mathbf{F}}^T, \quad \eta = \eta_E - \frac{\varpi}{\rho} \left\{ \partial_{\theta}(\bar{f}) - [\bar{\mathbf{F}}^T \partial_{\bar{\mathbf{F}}}(\bar{f})\mathbf{U}^{*-1}] : \frac{d\mathbf{U}^*}{d\theta} \right\}. \quad (133)$$

Isotropic constraints that are invariant to rigid body motions: The type of constraint considered here can be written as

$$f(I_1, I_2, I_3, \theta) = 0, \quad (134)$$

where $I_1 = \text{tr}(\mathbf{U})$, $I_2 = \text{tr}(\mathbf{U}^2)$, and $I_3 = \text{tr}(\mathbf{U}^3)$ are the isotropic invariants of \mathbf{U} . The time derivative of the constraint function is given by

$$\dot{f} = \frac{\partial f}{\partial I_1} \dot{I}_1 + \frac{\partial f}{\partial I_2} \dot{I}_2 + \frac{\partial f}{\partial I_3} \dot{I}_3 + \frac{\partial f}{\partial \theta} \dot{\theta}, \quad (135)$$

where $\dot{I}_1 = \mathbf{R} : \dot{\mathbf{F}}$, $\dot{I}_2 = 2\mathbf{F} : \dot{\mathbf{F}}$, $\dot{I}_3 = 3(\mathbf{V}\mathbf{F}) : \dot{\mathbf{F}}$. Substitution and reorganization yields

$$\dot{f} = \left[\frac{\partial f}{\partial I_1} \mathbf{R} + 2 \frac{\partial f}{\partial I_2} \mathbf{F} + 3 \frac{\partial f}{\partial I_3} \mathbf{V}\mathbf{F} \right] : \dot{\mathbf{F}} + \frac{\partial f}{\partial \theta} \dot{\theta}. \quad (136)$$

Therefore, the two partial derivatives are

$$\partial_{\mathbf{F}}(f) = \frac{\partial f}{\partial I_1} \mathbf{R} + 2 \frac{\partial f}{\partial I_2} \mathbf{F} + 3 \frac{\partial f}{\partial I_3} \mathbf{V}\mathbf{F}, \quad \partial_{\theta}(f) = \frac{\partial f}{\partial \theta}. \quad (137)$$

As a result, the expression for Cauchy stress and entropy for a general isotropic constraint that is invariant to rigid body motions and does not include temperature gradients is

$$\mathbf{T}^T = \mathbf{T}_E^T + \varpi \left[\frac{\partial f}{\partial I_1} \mathbf{V} + 2 \frac{\partial f}{\partial I_2} \mathbf{V}^2 + 3 \frac{\partial f}{\partial I_3} \mathbf{V}^3 \right], \quad \eta = \eta_E - \frac{\varpi}{\rho} \frac{\partial f}{\partial \theta}. \quad (138)$$

A similar expression can be found if the constraint is written as $f(I, II, III, \theta) = 0$, where $I = \text{tr}(\mathbf{U})$, $II = [\text{tr}^2(\mathbf{U}) - \text{tr}(\mathbf{U}^2)]/2$, and $III = \det(\mathbf{U})$ are an alternate set of isotropic invariants of \mathbf{U} . Using the relations $\dot{I} = \mathbf{R} : \dot{\mathbf{F}}$, $\dot{II} = [\text{tr}(\mathbf{U}\mathbf{R} - \mathbf{F})] : \dot{\mathbf{F}}$, and $\dot{III} = \det(\mathbf{U})\mathbf{F}^{-T} : \dot{\mathbf{F}}$, one obtains

$$\mathbf{T}^T = \mathbf{T}_E^T + \varpi \left\{ \det(\mathbf{V}) \frac{\partial f}{\partial III} \mathbf{I} + \left[\frac{\partial f}{\partial I} + \text{tr}(\mathbf{V}) \frac{\partial f}{\partial II} \right] \mathbf{V} - \frac{\partial f}{\partial II} \mathbf{V}^2 \right\}. \quad (139)$$

This could also be shown using the Cayley–Hamilton theorem. A commonly used alternate method is based on the isotropic invariants of \mathbf{C} . For example, consider $f(I_1^*, I_2^*, I_3^*, \theta) = 0$, where $I_1^* = \text{tr}(\mathbf{C})$, $I_2^* = \text{tr}(\mathbf{C}^2)$, and $I_3^* = \text{tr}(\mathbf{C}^3)$. Using $\dot{I}_1^* = 2\mathbf{F} : \dot{\mathbf{F}}$, $\dot{I}_2^* = 4(\mathbf{F}\mathbf{C}) : \dot{\mathbf{F}}$, and $\dot{I}_3^* = 6(\mathbf{F}\mathbf{C}^2) : \dot{\mathbf{F}}$, one obtains

$$\mathbf{T}^T = \mathbf{T}_E^T + 2\varpi \left[\frac{\partial f}{\partial I_1^*} \mathbf{B} + 2 \frac{\partial f}{\partial I_2^*} \mathbf{B}^2 + 3 \frac{\partial f}{\partial I_3^*} \mathbf{B}^3 \right]. \quad (140)$$

Another method is based on the alternate isotropic invariants of \mathbf{C} . For this case $f(I^*, II^*, III^*, \theta) = 0$, where $I^* = \text{tr}(\mathbf{C})$, $II^* = [\text{tr}^2(\mathbf{C}) - \text{tr}(\mathbf{C}^2)]/2$, and $III^* = \det(\mathbf{C})$. Using $\dot{I}^* = 2\mathbf{F} : \dot{\mathbf{F}}$, $\dot{II}^* = 2[\text{tr}(\mathbf{C})\mathbf{F} - \mathbf{F}\mathbf{C}] : \dot{\mathbf{F}}$, and $\dot{III}^* = 2\det(\mathbf{C})(\mathbf{F}\mathbf{C}^{-1}) : \dot{\mathbf{F}}$, one obtains

$$\mathbf{T}^T = \mathbf{T}_E^T + 2\varpi \left\{ \det(\mathbf{C}) \frac{\partial f}{\partial III^*} \mathbf{I} + \left[\frac{\partial f}{\partial I^*} + \text{tr}(\mathbf{C}) \frac{\partial f}{\partial II^*} \right] \mathbf{B} - \frac{\partial f}{\partial II^*} \mathbf{B}^2 \right\}. \quad (141)$$

Zero temperature gradient normal to a material surface: The constraint considered here is the case where the temperature gradient across a material surface \mathcal{S} in the current configuration is zero. For any vector \mathbf{n} normal to \mathcal{S} , this constraint can be written as

$$\mathbf{g} \circ \mathbf{n} = 0. \quad (142)$$

Let \mathcal{S}_o denote the same material surface, but in the reference configuration, and let $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ denote two orthogonal unit vectors on the surface \mathcal{S}_o so that $\hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2$ is a unit vector perpendicular to the surface \mathcal{S}_o . The vectors $\mathbf{u}_1 = \mathbf{F}\hat{\mathbf{e}}_1$ and $\mathbf{u}_2 = \mathbf{F}\hat{\mathbf{e}}_2$ are two vectors on the surface \mathcal{S} , and can be used to calculate the normal \mathbf{n} to this surface through the relation $\mathbf{n} = \mathbf{u}_1 \times \mathbf{u}_2 = (\mathbf{F}\hat{\mathbf{e}}_1) \times (\mathbf{F}\hat{\mathbf{e}}_2) = \det(\mathbf{F})\hat{\mathbf{e}}_3\mathbf{F}^{-1}$, which results after using the relation $(\mathbf{F}\mathbf{a}) \times (\mathbf{F}\mathbf{b}) = \det(\mathbf{F})(\mathbf{a} \times \mathbf{b})\mathbf{F}^{-1}$ for any two vectors \mathbf{a} and \mathbf{b} . Obviously, such a vector \mathbf{n} is not necessarily a unit vector since $\mathbf{n} \circ \mathbf{n} = \det^2(\mathbf{F})\hat{\mathbf{e}}_3 \circ (\mathbf{C}^{-1}\hat{\mathbf{e}}_3)$ need not equal unity. Adding to the expression for \mathbf{n} the relation $\mathbf{g} = \mathbf{G}\mathbf{F}^{-1}$ and substitution into the constraint equation yields the relation

$$f = \mathbf{G} \circ (\mathbf{C}^{-1}\hat{\mathbf{e}}_3) = 0, \quad (143)$$

where the original constraint has been divided by $\det(\mathbf{F})$. It can be shown that the partial derivatives of this constraint equation become

$$\partial_{\mathbf{F}}(f) = \mathbf{F}^{-T}(\hat{\mathbf{e}}_3 \otimes \mathbf{G} + \mathbf{G} \otimes \hat{\mathbf{e}}_3)\mathbf{C}^{-1}, \quad \partial_\theta(f) = 0, \quad \partial_{\mathbf{G}}(f) = \mathbf{C}^{-1}\hat{\mathbf{e}}_3. \quad (144)$$

Therefore, the expressions for Cauchy stress and entropy can be written as

$$\mathbf{T}^T = \mathbf{T}_E^T + \varpi \mathbf{F}^{-T}(\hat{\mathbf{e}}_3 \otimes \mathbf{G} + \mathbf{G} \otimes \hat{\mathbf{e}}_3)\mathbf{F}^{-1}, \quad \eta = \eta_E. \quad (145)$$

Using the relations given above, the Cauchy stress can also be expressed as

$$\mathbf{T}^T = \mathbf{T}_E^T + \frac{\varpi}{\det(\mathbf{F})}(\mathbf{n} \otimes \mathbf{g} + \mathbf{g} \otimes \mathbf{n}). \quad (146)$$

Using the third method to decouple the system one obtains

$$\begin{aligned} \mathbf{T}_E^{(3)T} &= \mathbf{T}_E^T - \frac{\hat{\mathbf{g}} \circ (\mathbf{T}_E^T \hat{\mathbf{n}}) + \hat{\mathbf{n}} \circ (\mathbf{T}_E^T \hat{\mathbf{g}})}{2}(\hat{\mathbf{n}} \otimes \hat{\mathbf{g}} + \hat{\mathbf{g}} \otimes \hat{\mathbf{n}}), \\ \mathbf{T}_R^{(3)T} &= \frac{\hat{\mathbf{g}} \circ (\mathbf{T}^T \hat{\mathbf{n}}) + \hat{\mathbf{n}} \circ (\mathbf{T}^T \hat{\mathbf{g}})}{2}(\hat{\mathbf{n}} \otimes \hat{\mathbf{g}} + \hat{\mathbf{g}} \otimes \hat{\mathbf{n}}), \end{aligned} \quad (147)$$

where $\hat{\mathbf{n}} = \mathbf{n}/|\mathbf{n}|$ and $\hat{\mathbf{g}} = \mathbf{g}/|\mathbf{g}|$ are unit vectors along \mathbf{n} and \mathbf{g} , respectively. As is obviously clear, the term $[\hat{\mathbf{g}} \circ (\mathbf{T}^T \hat{\mathbf{n}}) + \hat{\mathbf{n}} \circ (\mathbf{T}^T \hat{\mathbf{g}})]/2$ is the shear stress on the surface with normal $\hat{\mathbf{n}}$ along the direction $\hat{\mathbf{g}}$. An additional result for this constraint is obtained from manipulating the condition $\mathbf{q} \circ \mathbf{g} \leq 0$. Note that given any arbitrary temperature gradient \mathbf{g}^* , one can construct a temperature gradient $\mathbf{g} = \mathbf{g}^* - \gamma \hat{\mathbf{n}}$ that satisfies the constraint $\mathbf{g} \circ \hat{\mathbf{n}} = 0$ by selecting $\gamma = \mathbf{g}^* \circ \hat{\mathbf{n}}$. Substitution of \mathbf{g} into $\mathbf{q} \circ \mathbf{g} \leq 0$ and reorganization gives

$$(\mathbf{q} - \mathbf{q} \circ \hat{\mathbf{n}} \hat{\mathbf{n}}) \circ \mathbf{g}^* \leq 0, \quad (148)$$

which states that one can change the component of heat flux \mathbf{q} along $\hat{\mathbf{n}}$ by any amount without violating the Clausius–Duhem inequality. Similar to the inextensibility constraint, even though not a direct conclusion of this equation, one may interpret this as stating that if the temperature gradient along a given direction is forced to be zero, then the heat flux along that direction will take any value required to satisfy the balance laws and boundary conditions. In view of (41), it obviously follows that if the constitutive function for the free-energy is independent of \mathbf{G} , this requires that ϖ be zero and results in $\mathbf{T} = \mathbf{T}_E$.

8. EXAMPLES OF MULTIPLE CONSTRAINTS

In the following examples thermoelastic response in the presence of more than one constraint is considered. In each case the third method of grouping terms will be used to illustrate the resulting forms. For this method $\mathbf{T} = \mathbf{T}_E^{(3)} + \mathbf{T}_R^{(3)}$ with the property $\mathbf{T}_E^{(3)} : \mathbf{T}_R^{(3)} = 0$ so that $\mathbf{T} : \mathbf{T}_E^{(3)} = \mathbf{T}_E^{(3)} : \mathbf{T}_E^{(3)}$ and $\mathbf{T} : \mathbf{T}_R^{(3)} = \mathbf{T}_R^{(3)} : \mathbf{T}_R^{(3)}$. A number of additional examples related to the Bell constraint in combination with inextensibility are included in the Appendix.

Isothermally incompressible and inextensible: A material that is both isothermally incompressible and isothermally inextensible along the direction $\hat{\mathbf{h}}_o$ in the reference configuration is described by the two constraints

$$f_1 = \det(\mathbf{F}) - J^*(\theta) = 0, \quad (149)$$

$$f_2 = \hat{\mathbf{h}}_o \circ (\mathbf{C}\hat{\mathbf{h}}_o) - \lambda^2(\theta) = 0, \quad (150)$$

where J^* and λ are as described above for single constraints. The associated partial derivatives are

$$\partial_{\mathbf{F}}(f_1) = \det(\mathbf{F})\mathbf{F}^{-T}, \quad \partial_{\theta}(f_1) = -\frac{dJ^*}{d\theta}, \quad (151)$$

$$\partial_{\mathbf{F}}(f_2) = 2(\mathbf{F}\hat{\mathbf{h}}_o) \otimes \hat{\mathbf{h}}_o, \quad \partial_{\theta}(f_2) = -2\lambda \frac{d\lambda}{d\theta}. \quad (152)$$

This results in the expressions for Cauchy stress and entropy given as

$$\mathbf{T}^T = \mathbf{T}_E^T + \varpi_1 J^* \mathbf{I} + 2\varpi_2 (\mathbf{F}\hat{\mathbf{h}}_o) \otimes (\mathbf{F}\hat{\mathbf{h}}_o), \quad (153)$$

$$\eta = \eta_E + \frac{\varpi_1}{\rho} \frac{dJ^*}{d\theta} + \frac{2\varpi_2}{\rho} \lambda \frac{d\lambda}{d\theta}. \quad (154)$$

The matrix $[K]$ with components $K_{ij} = [\partial_{\mathbf{F}}(f_i)\mathbf{F}^T] : [\partial_{\mathbf{F}}(f_j)\mathbf{F}^T]$ is, therefore, given by

$$[K] = \begin{bmatrix} 3J^{*2} & 2J^*\lambda^2 \\ 2J^*\lambda^2 & 4\lambda^4 \end{bmatrix}. \quad (155)$$

The inverse of this matrix is given by

$$[K]^{-1} = \frac{1}{8J^{*2}\lambda^4} \begin{bmatrix} 4\lambda & -2J^*\lambda^2 \\ -2J^*\lambda^2 & 3J^{*2} \end{bmatrix}, \quad (156)$$

which results in the expressions

$$\begin{aligned} \mathbf{T}_E^{(3)T} &= \mathbf{T}_E^T + \frac{1}{2} \left[-\text{tr}(\mathbf{T}_E) + \hat{\mathbf{h}} \circ (\mathbf{T}_E^T \hat{\mathbf{h}}) \right] \mathbf{I} \\ &+ \frac{1}{2} \left[\text{tr}(\mathbf{T}_E) - 3\hat{\mathbf{h}} \circ (\mathbf{T}_E^T \hat{\mathbf{h}}) \right] \hat{\mathbf{h}} \otimes \hat{\mathbf{h}}, \end{aligned} \quad (157)$$

$$\mathbf{T}_R^{(3)T} = \frac{1}{2} \left[\text{tr}(\mathbf{T}) - \hat{\mathbf{h}} \circ (\mathbf{T}^T \hat{\mathbf{h}}) \right] \mathbf{I} + \frac{1}{2} \left[-\text{tr}(\mathbf{T}) + 3\hat{\mathbf{h}} \circ (\mathbf{T}^T \hat{\mathbf{h}}) \right] \hat{\mathbf{h}} \otimes \hat{\mathbf{h}}, \quad (158)$$

$$\begin{aligned}\eta_E^{(3)} &= \eta_E + \frac{1}{2\rho} \left[-\text{tr}(\mathbf{T}_E) + \hat{\mathbf{h}} \circ (\mathbf{T}_E^T \hat{\mathbf{h}}) \right] \frac{dJ^*}{d\theta} \\ &+ \frac{1}{\rho} \left[\text{tr}(\mathbf{T}_E) - 3\hat{\mathbf{h}} \circ (\mathbf{T}_E^T \hat{\mathbf{h}}) \right] \lambda \frac{d\lambda}{d\theta},\end{aligned}\quad (159)$$

$$\eta_R^{(3)} = \frac{1}{2\rho} \left[\text{tr}(\mathbf{T}) - \hat{\mathbf{h}} \circ (\mathbf{T}^T \hat{\mathbf{h}}) \right] \frac{dJ^*}{d\theta} + \frac{1}{\rho} \left[-\text{tr}(\mathbf{T}) + 3\hat{\mathbf{h}} \circ (\mathbf{T}^T \hat{\mathbf{h}}) \right] \lambda \frac{d\lambda}{d\theta}, \quad (160)$$

where $\hat{\mathbf{h}} = \mathbf{F}\hat{\mathbf{h}}_o/\lambda$. As the reader will note, the complexity added by this grouping of terms results in the orthogonality of $\mathbf{T}_E^{(3)}$ and $\mathbf{T}_R^{(3)}$ (i.e., $\mathbf{T}_E^{(3)} : \mathbf{T}_R^{(3)} = 0$).

Isothermally incompressible and Bell constrained: A material that is both isothermally incompressible and isothermally Bell constrained is described by the two constraints

$$f_1 = \det(\mathbf{F}) - J^*(\theta) = 0, \quad (161)$$

$$f_2 = \text{tr}(\mathbf{V}) - 3J^{*1/3}(\theta) = 0, \quad (162)$$

where J^* and λ are as described above for single constraints and assuming isotropic thermal expansion. Even though it has been shown by Beatty and Hayes [23] that these constraints are incompatible at all points other than the initial unloaded configuration, let us follow through the steps to see where this incompatibility is exposed. The associated partial derivatives are

$$\partial_{\mathbf{F}}(f_1) = \det(\mathbf{F})\mathbf{F}^{-T}, \quad \partial_{\theta}(f_1) = -\frac{dJ^*}{d\theta}, \quad (163)$$

$$\partial_{\mathbf{F}}(f_2) = \mathbf{R}, \quad \partial_{\theta}(f_2) = -J^{*-2/3} \frac{dJ^*}{d\theta}. \quad (164)$$

This results in the expressions for Cauchy stress and entropy as

$$\mathbf{T}^T = \mathbf{T}_E^T + \varpi_1 J^* \mathbf{I} + \varpi_2 \mathbf{V}, \quad (165)$$

$$\eta = \eta_E + \frac{\varpi_1}{\rho} \frac{dJ^*}{d\theta} + \frac{\varpi_2}{\rho} J^{*-2/3} \frac{dJ^*}{d\theta}. \quad (166)$$

The matrix $[K]$ with components $K_{ij} = [\partial_{\mathbf{F}}(f_i)\mathbf{F}^T] : [\partial_{\mathbf{F}}(f_j)\mathbf{F}^T]$ is, therefore, given by

$$[K] = \begin{bmatrix} 3J^{*2} & J^* \text{tr}(\mathbf{V}) \\ J^* \text{tr}(\mathbf{V}) & \text{tr}(\mathbf{B}) \end{bmatrix}. \quad (167)$$

The determinant of $[K]$ is given by $\det([K]) = 3J^{*2}\text{tr}(\mathbf{B}) - J^{*2}\text{tr}^2(\mathbf{V})$ which is zero for the reference configuration (i.e., for $\mathbf{F} = \mathbf{I}$). Therefore, *the Cauchy stress cannot be separated into two parts which are orthogonal*. This is the first indication of a bigger problem. It turns out that even though these two constraints are compatible for $\mathbf{V} = \mathbf{I}$ at the reference configuration and temperature, they are not compatible at any other deformations at the reference temperature. That is, *the system becomes locked* under isothermal conditions.

Let us return to the original process and recall that the process that was developed required that the matrix $[A]$ with components $A_{ij} = [\partial_{\mathcal{L}}(f_i)] \circ [\partial_{\mathcal{L}}(f_j)]$ must be invertible. For these two constraints one has

$$[A] = \begin{bmatrix} J^{*2} \text{tr}(\mathbf{C}^{-1}) + \left(\frac{dJ^*}{d\theta}\right)^2 & J^* \text{tr}(\mathbf{V}^{-1}) + J^{*-2/3} \left(\frac{dJ^*}{d\theta}\right)^2 \\ J^* \text{tr}(\mathbf{V}^{-1}) + J^{*-2/3} \left(\frac{dJ^*}{d\theta}\right)^2 & 3 + (J^{*-2/3} \frac{dJ^*}{d\theta})^2 \end{bmatrix}. \quad (168)$$

As can be seen $\det[A] = 0$ for $\mathbf{V} = \mathbf{I}$ and, therefore, $[A]$ is not invertible at the reference configuration. If $[A]$ cannot be inverted, one cannot find the needed α_i from (70) to properly decouple the terms in the Clausius–Duhem inequality and one cannot obtain the results given in (70)–(74). This is fully consistent with the observation of Beatty and Hayes [23] that these constraints are incompatible at all points other than for $\mathbf{V} = \mathbf{I}$.

Isothermally inextensible along two directions: A material that is isothermally inextensible along the two directions $\hat{\mathbf{h}}_o$ and $\hat{\mathbf{m}}_o$ in the reference configuration is described by the two constraints

$$f_1 = \hat{\mathbf{h}}_o \circ (\mathbf{C}\hat{\mathbf{h}}_o) - \lambda_1^2(\theta) = 0, \quad (169)$$

$$f_2 = \hat{\mathbf{m}}_o \circ (\mathbf{C}\hat{\mathbf{m}}_o) - \lambda_2^2(\theta) = 0, \quad (170)$$

where λ_1 and λ_2 are the stretches associated with thermal expansions along the respective constraint directions. The associated partial derivatives are

$$\partial_{\mathbf{F}}(f_1) = 2(\mathbf{F}\hat{\mathbf{h}}_o) \otimes \hat{\mathbf{h}}_o, \quad \partial_{\theta}(f_1) = -2\lambda_1 \frac{d\lambda_1}{d\theta}, \quad (171)$$

$$\partial_{\mathbf{F}}(f_2) = 2(\mathbf{F}\hat{\mathbf{m}}_o) \otimes \hat{\mathbf{m}}_o, \quad \partial_{\theta}(f_2) = -2\lambda_2 \frac{d\lambda_2}{d\theta}. \quad (172)$$

This results in the expressions for Cauchy stress and entropy as

$$\mathbf{T}^T = \mathbf{T}_E^T + 2\varpi_1(\mathbf{F}\hat{\mathbf{h}}_o) \otimes (\mathbf{F}\hat{\mathbf{h}}_o) + 2\varpi_2(\mathbf{F}\hat{\mathbf{m}}_o) \otimes (\mathbf{F}\hat{\mathbf{m}}_o), \quad (173)$$

$$\eta = \eta_E + \frac{2\varpi_1}{\rho} \lambda_1 \frac{d\lambda_1}{d\theta} + \frac{2\varpi_2}{\rho} \lambda_2 \frac{d\lambda_2}{d\theta}. \quad (174)$$

The matrix $[K]$ with components $K_{ij} = [\partial_{\mathbf{F}}(f_i)\mathbf{F}^T] : [\partial_{\mathbf{F}}(f_j)\mathbf{F}^T]$ is, therefore, given by

$$[K] = \begin{bmatrix} 4\lambda_1^4 & 4\lambda_1^2\lambda_2^2(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2 \\ 4\lambda_1^2\lambda_2^2(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2 & 4\lambda_2^4 \end{bmatrix}, \quad (175)$$

where $\hat{\mathbf{h}} = \mathbf{F}\hat{\mathbf{h}}_o/\lambda_1$ and $\hat{\mathbf{m}} = \mathbf{F}\hat{\mathbf{m}}_o/\lambda_2$. The inverse of this matrix is given by

$$[K]^{-1} = \frac{1}{4\lambda_1^4\lambda_2^4[1 - (\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^4]} \begin{bmatrix} \lambda_2^4 & -\lambda_1^2\lambda_2^2(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2 \\ -\lambda_1^2\lambda_2^2(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2 & \lambda_1^4 \end{bmatrix}, \quad (176)$$

which is clearly singular when the two directions $\hat{\mathbf{h}}$ and $\hat{\mathbf{m}}$ are parallel. The expression for stress and entropy can now be written as

$$\begin{aligned} \mathbf{T}_E^{(3)T} &= \mathbf{T}_E^T - \frac{\hat{\mathbf{h}} \circ (\mathbf{T}_E \hat{\mathbf{h}}) - \hat{\mathbf{m}} \circ (\mathbf{T}_E \hat{\mathbf{m}})(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2}{1 - (\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^4} \hat{\mathbf{h}} \otimes \hat{\mathbf{h}} \\ &\quad - \frac{\hat{\mathbf{m}} \circ (\mathbf{T}_E \hat{\mathbf{m}}) - \hat{\mathbf{h}} \circ (\mathbf{T}_E \hat{\mathbf{h}})(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2}{1 - (\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^4} \hat{\mathbf{m}} \otimes \hat{\mathbf{m}}, \end{aligned} \quad (177)$$

$$\begin{aligned} \mathbf{T}_R^{(3)T} &= \frac{\hat{\mathbf{h}} \circ (\mathbf{T}\hat{\mathbf{h}}) - \hat{\mathbf{m}} \circ (\mathbf{T}\hat{\mathbf{m}})(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2}{1 - (\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^4} \hat{\mathbf{h}} \otimes \hat{\mathbf{h}} \\ &\quad + \frac{\hat{\mathbf{m}} \circ (\mathbf{T}\hat{\mathbf{m}}) - \hat{\mathbf{h}} \circ (\mathbf{T}\hat{\mathbf{h}})(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2}{1 - (\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^4} \hat{\mathbf{m}} \otimes \hat{\mathbf{m}}, \end{aligned} \quad (178)$$

$$\begin{aligned} \eta_E^{(3)} &= \eta_E - \frac{\hat{\mathbf{h}} \circ (\mathbf{T}_E \hat{\mathbf{h}}) - \hat{\mathbf{m}} \circ (\mathbf{T}_E \hat{\mathbf{m}})(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2}{\rho\lambda_1 [1 - (\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^4]} \frac{d\lambda_1}{d\theta} \\ &\quad - \frac{\hat{\mathbf{m}} \circ (\mathbf{T}_E \hat{\mathbf{m}}) - \hat{\mathbf{h}} \circ (\mathbf{T}_E \hat{\mathbf{h}})(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2}{\rho\lambda_2 [1 - (\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^4]} \frac{d\lambda_3}{d\theta}, \end{aligned} \quad (179)$$

$$\begin{aligned} \eta_R^{(3)} &= \frac{\hat{\mathbf{h}} \circ (\mathbf{T}\hat{\mathbf{h}}) - \hat{\mathbf{m}} \circ (\mathbf{T}\hat{\mathbf{m}})(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2}{\rho\lambda_1 [1 - (\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^4]} \frac{d\lambda_2}{d\theta} \\ &\quad + \frac{\hat{\mathbf{m}} \circ (\mathbf{T}\hat{\mathbf{m}}) - \hat{\mathbf{h}} \circ (\mathbf{T}\hat{\mathbf{h}})(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2}{\rho\lambda_2 [1 - (\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^4]} \frac{d\lambda_3}{d\theta}. \end{aligned} \quad (180)$$

Isothermally inextensible along three directions: A material that is isothermally inextensible along the three directions $\hat{\mathbf{h}}_o$, $\hat{\mathbf{m}}_o$, and $\hat{\mathbf{n}}_o$ in the reference configuration is described by the three constraints

$$f_1 = \hat{\mathbf{h}}_o \circ (\mathbf{C}\hat{\mathbf{h}}_o) - \lambda_1^2(\theta) = 0, \quad (181)$$

$$f_2 = \hat{\mathbf{m}}_o \circ (\mathbf{C}\hat{\mathbf{m}}_o) - \lambda_2^2(\theta) = 0, \quad (182)$$

$$f_3 = \hat{\mathbf{n}}_o \circ (\mathbf{C}\hat{\mathbf{n}}_o) - \lambda_3^2(\theta) = 0, \quad (183)$$

where λ_1 , λ_2 and λ_3 are the stretches associated with thermal expansions along the respective constraint directions. The associated partial derivatives are

$$\partial_{\mathbf{F}}(f_1) = 2(\mathbf{F}\hat{\mathbf{h}}_o) \otimes \hat{\mathbf{h}}_o, \quad \partial_{\theta}(f_1) = -2\lambda_1 \frac{d\lambda_1}{d\theta}, \quad (184)$$

$$\partial_{\mathbf{F}}(f_2) = 2(\mathbf{F}\hat{\mathbf{m}}_o) \otimes \hat{\mathbf{m}}_o, \quad \partial_{\theta}(f_2) = -2\lambda_2 \frac{d\lambda_2}{d\theta}, \quad (185)$$

$$\partial_{\mathbf{F}}(f_3) = 2(\mathbf{F}\hat{\mathbf{n}}_o) \otimes \hat{\mathbf{n}}_o, \quad \partial_{\theta}(f_3) = -2\lambda_3 \frac{d\lambda_3}{d\theta}. \quad (186)$$

This results in the expressions for Cauchy stress and entropy as

$$\mathbf{T}^T = \mathbf{T}_E^T + 2\varpi_1(\mathbf{F}\hat{\mathbf{h}}_o) \otimes (\mathbf{F}\hat{\mathbf{h}}_o) + 2\varpi_2(\mathbf{F}\hat{\mathbf{m}}_o) \otimes (\mathbf{F}\hat{\mathbf{m}}_o) + 2\varpi_3(\mathbf{F}\hat{\mathbf{n}}_o) \otimes (\mathbf{F}\hat{\mathbf{n}}_o), \quad (187)$$

$$\eta = \eta_E + \frac{2\varpi_1}{\rho} \lambda_1 \frac{d\lambda_1}{d\theta} + \frac{2\varpi_2}{\rho} \lambda_2 \frac{d\lambda_2}{d\theta} + \frac{2\varpi_3}{\rho} \lambda_3 \frac{d\lambda_3}{d\theta}. \quad (188)$$

The matrix $[K]$ with components $K_{ij} = [\partial_{\mathbf{F}}(f_i)\mathbf{F}^T] : [\partial_{\mathbf{F}}(f_j)\mathbf{F}^T]$ is, therefore, given by

$$[K] = 4 \begin{bmatrix} \lambda_1^4 & \lambda_1^2 \lambda_2^2 (\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2 & \lambda_1^2 \lambda_3^2 (\hat{\mathbf{h}} \circ \hat{\mathbf{n}})^2 \\ \lambda_1^2 \lambda_2^2 (\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2 & \lambda_2^4 & \lambda_2^2 \lambda_3^2 (\hat{\mathbf{m}} \circ \hat{\mathbf{n}})^2 \\ \lambda_1^2 \lambda_3^2 (\hat{\mathbf{h}} \circ \hat{\mathbf{n}})^2 & \lambda_2^2 \lambda_3^2 (\hat{\mathbf{m}} \circ \hat{\mathbf{n}})^2 & \lambda_3^4 \end{bmatrix}, \quad (189)$$

where $\hat{\mathbf{h}} = \mathbf{F}\hat{\mathbf{h}}_o/\lambda_1$, $\hat{\mathbf{m}} = \mathbf{F}\hat{\mathbf{m}}_o/\lambda_2$ and $\hat{\mathbf{n}} = \mathbf{F}\hat{\mathbf{n}}_o/\lambda_3$. The inverse of this matrix is given by

$$[K]^{-1} = \frac{1}{4D} \times \begin{bmatrix} \frac{(\hat{\mathbf{m}} \circ \hat{\mathbf{n}})^4 - 1}{\lambda_1^4} & \frac{(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2 - (\hat{\mathbf{h}} \circ \hat{\mathbf{n}})^2 (\hat{\mathbf{m}} \circ \hat{\mathbf{n}})^2}{\lambda_1^2 \lambda_2^2} & \frac{(\hat{\mathbf{h}} \circ \hat{\mathbf{n}})^2 - (\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2 (\hat{\mathbf{m}} \circ \hat{\mathbf{n}})^2}{\lambda_1^2 \lambda_3^2} \\ \frac{(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2 - (\hat{\mathbf{h}} \circ \hat{\mathbf{n}})^2 (\hat{\mathbf{m}} \circ \hat{\mathbf{n}})^2}{\lambda_1^2 \lambda_2^2} & \frac{(\hat{\mathbf{h}} \circ \hat{\mathbf{n}})^4 - 1}{\lambda_2^4} & \frac{(\hat{\mathbf{m}} \circ \hat{\mathbf{n}})^2 - (\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2 (\hat{\mathbf{h}} \circ \hat{\mathbf{n}})^2}{\lambda_2^2 \lambda_3^2} \\ \frac{(\hat{\mathbf{h}} \circ \hat{\mathbf{n}})^2 - (\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2 (\hat{\mathbf{m}} \circ \hat{\mathbf{n}})^2}{\lambda_1^2 \lambda_3^2} & \frac{(\hat{\mathbf{m}} \circ \hat{\mathbf{n}})^2 - (\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2 (\hat{\mathbf{h}} \circ \hat{\mathbf{n}})^2}{\lambda_2^2 \lambda_3^2} & \frac{(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^4 - 1}{\lambda_3^4} \end{bmatrix}, \quad (190)$$

where

$$D = (\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^4 + (\hat{\mathbf{h}} \circ \hat{\mathbf{n}})^4 + (\hat{\mathbf{m}} \circ \hat{\mathbf{n}})^4 - 2(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2 (\hat{\mathbf{h}} \circ \hat{\mathbf{n}})^2 (\hat{\mathbf{m}} \circ \hat{\mathbf{n}})^2 - 1. \quad (191)$$

It is clear that $[K]$ is singular when any two of the three directions $\hat{\mathbf{h}}$, $\hat{\mathbf{m}}$, and $\hat{\mathbf{n}}$ are parallel and orthogonal to the third direction. From $[K]^{-1}$ one can calculate $\bar{\varpi}_i^{(3)}$ and $\varpi_i^{*(3)}$ as follows:

$$\begin{aligned} \bar{\varpi}_1^{(3)} &= -\frac{1}{2\lambda_1^2 D} \left\{ \hat{\mathbf{h}} \circ (\mathbf{T}_E \hat{\mathbf{h}}) [(\hat{\mathbf{m}} \circ \hat{\mathbf{n}})^4 - 1] + \hat{\mathbf{m}} \circ (\mathbf{T}_E \hat{\mathbf{m}}) [(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2 - (\hat{\mathbf{h}} \circ \hat{\mathbf{n}})^2 \right. \\ &\quad \times (\hat{\mathbf{m}} \circ \hat{\mathbf{n}})^2] + \hat{\mathbf{n}} \circ (\mathbf{T}_E \hat{\mathbf{n}}) [(\hat{\mathbf{h}} \circ \hat{\mathbf{n}})^2 - (\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2 (\hat{\mathbf{m}} \circ \hat{\mathbf{n}})^2] \left. \right\}, \quad (192) \end{aligned}$$

$$\begin{aligned} \bar{\varpi}_2^{(3)} &= -\frac{1}{2\lambda_2^2 D} \left\{ \hat{\mathbf{h}} \circ (\mathbf{T}_E \hat{\mathbf{h}}) [(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2 - (\hat{\mathbf{h}} \circ \hat{\mathbf{n}})^2 (\hat{\mathbf{m}} \circ \hat{\mathbf{n}})^2] + \hat{\mathbf{m}} \circ (\mathbf{T}_E \hat{\mathbf{m}}) \right. \\ &\quad \times [(\hat{\mathbf{h}} \circ \hat{\mathbf{n}})^4 - 1] + \hat{\mathbf{n}} \circ (\mathbf{T}_E \hat{\mathbf{n}}) [(\hat{\mathbf{m}} \circ \hat{\mathbf{n}})^2 - (\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2 (\hat{\mathbf{h}} \circ \hat{\mathbf{n}})^2] \left. \right\}, \quad (193) \end{aligned}$$

$$\begin{aligned} \bar{\varpi}_3^{(3)} &= -\frac{1}{2\lambda_3^2 D} \left\{ \hat{\mathbf{h}} \circ (\mathbf{T}_E \hat{\mathbf{h}}) \left[(\hat{\mathbf{h}} \circ \hat{\mathbf{n}})^2 - (\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2 (\hat{\mathbf{m}} \circ \hat{\mathbf{n}})^2 \right] + \hat{\mathbf{m}} \circ (\mathbf{T}_E \hat{\mathbf{m}}) \right. \\ &\quad \times \left. \left[((\hat{\mathbf{m}} \circ \hat{\mathbf{n}})^2 - (\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2 (\hat{\mathbf{h}} \circ \hat{\mathbf{n}})^2) + \hat{\mathbf{n}} \circ (\mathbf{T}_E \hat{\mathbf{n}}) \left[(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^4 - 1 \right] \right] \right\}, \end{aligned} \quad (194)$$

$$\begin{aligned} \varpi_1^{*(3)} &= \frac{1}{2\lambda_1^2 D} \left\{ \hat{\mathbf{h}} \circ (\mathbf{T}\hat{\mathbf{h}}) \left[(\hat{\mathbf{m}} \circ \hat{\mathbf{n}})^4 - 1 \right] + \hat{\mathbf{m}} \circ (\mathbf{T}\hat{\mathbf{m}}) \left[(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2 - (\hat{\mathbf{h}} \circ \hat{\mathbf{n}})^2 \right. \right. \\ &\quad \times \left. \left. (\hat{\mathbf{m}} \circ \hat{\mathbf{n}})^2 \right] + \hat{\mathbf{n}} \circ (\mathbf{T}\hat{\mathbf{n}}) \left[(\hat{\mathbf{h}} \circ \hat{\mathbf{n}})^2 - (\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2 (\hat{\mathbf{m}} \circ \hat{\mathbf{n}})^2 \right] \right\}, \end{aligned} \quad (195)$$

$$\begin{aligned} \varpi_2^{*(3)} &= \frac{1}{2\lambda_2^2 D} \left\{ \hat{\mathbf{h}} \circ (\mathbf{T}\hat{\mathbf{h}}) \left[(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2 - (\hat{\mathbf{h}} \circ \hat{\mathbf{n}})^2 (\hat{\mathbf{m}} \circ \hat{\mathbf{n}})^2 \right] + \hat{\mathbf{m}} \circ (\mathbf{T}\hat{\mathbf{m}}) \right. \\ &\quad \times \left. \left[(\hat{\mathbf{h}} \circ \hat{\mathbf{n}})^4 - 1 \right] + \hat{\mathbf{n}} \circ (\mathbf{T}\hat{\mathbf{n}}) \left[(\hat{\mathbf{m}} \circ \hat{\mathbf{n}})^2 - (\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2 (\hat{\mathbf{h}} \circ \hat{\mathbf{n}})^2 \right] \right\}, \end{aligned} \quad (196)$$

$$\begin{aligned} \varpi_3^{*(3)} &= \frac{1}{2\lambda_3^2 D} \left\{ \hat{\mathbf{h}} \circ (\mathbf{T}\hat{\mathbf{h}}) \left[(\hat{\mathbf{h}} \circ \hat{\mathbf{n}})^2 - (\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2 (\hat{\mathbf{m}} \circ \hat{\mathbf{n}})^2 \right] + \hat{\mathbf{m}} \circ (\mathbf{T}\hat{\mathbf{m}}) \right. \\ &\quad \times \left. \left[((\hat{\mathbf{m}} \circ \hat{\mathbf{n}})^2 - (\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2 (\hat{\mathbf{h}} \circ \hat{\mathbf{n}})^2) + \hat{\mathbf{n}} \circ (\mathbf{T}\hat{\mathbf{n}}) \left[(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^4 - 1 \right] \right] \right\}. \end{aligned} \quad (197)$$

The expressions for stress and entropy can now be written as

$$\mathbf{T}_E^{(3)T} = \mathbf{T}_E^T + 2\lambda_1 \bar{\varpi}_1^{(3)} \hat{\mathbf{h}} \otimes \hat{\mathbf{h}} + 2\lambda_2 \bar{\varpi}_2^{(3)} \hat{\mathbf{m}} \otimes \hat{\mathbf{m}} + 2\lambda_3 \bar{\varpi}_3^{(3)} \hat{\mathbf{n}} \otimes \hat{\mathbf{n}} \quad (198)$$

$$\mathbf{T}_R^{(3)T} = 2\lambda_1 \varpi_1^{*(3)} \hat{\mathbf{h}} \otimes \hat{\mathbf{h}} + 2\lambda_2 \varpi_2^{*(3)} \hat{\mathbf{m}} \otimes \hat{\mathbf{m}} + 2\lambda_3 \varpi_3^{*(3)} \hat{\mathbf{n}} \otimes \hat{\mathbf{n}}, \quad (199)$$

$$\eta_E^{(3)} = \eta_E + \frac{2\bar{\varpi}_1^{(3)} \lambda_1}{\rho} \frac{d\lambda_1}{d\theta} + \frac{2\bar{\varpi}_2^{(3)} \lambda_2}{\rho} \frac{d\lambda_2}{d\theta} + \frac{2\bar{\varpi}_3^{(3)} \lambda_3}{\rho} \frac{d\lambda_3}{d\theta}, \quad (200)$$

$$\eta_R^{(3)} = \frac{2\varpi_1^{*(3)} \lambda_1}{\rho} \frac{d\lambda_1}{d\theta} + \frac{2\varpi_2^{*(3)} \lambda_2}{\rho} \frac{d\lambda_2}{d\theta} + \frac{2\varpi_3^{*(3)} \lambda_3}{\rho} \frac{d\lambda_3}{d\theta}. \quad (201)$$

9. HISTORICAL NOTES

The following comments are a short review of the work on material constraints as relates to the current development.

Ericksen and Rivlin [4] in 1954 studied large elastic deformations with multiple kinematical constraints. Their development is based on satisfying the balance of energy, assuming the existence of a stored energy (strain energy) function. In doing so they note that in the absence of constraints, the balance of energy can be used to fully determine the form of the stress. In the presence of constraints it is shown by the same method used in this article (but applied to the balance of energy) that the stress is not fully determined by the balance of energy. This is due to the fact that all components of the velocity gradient, that both sides of their equation are multiplied by, can no longer be selected arbitrarily. The result is the introduction of an indeterminacy in the expression for stress.

Green and Adkins [6] in 1960 used virtual work as a starting point. This procedure results in equations identical to the balance of energy. They also point out that the strain energy function may be regarded as arbitrary up to the addition of a scalar function of the constraints.

Truesdell and Noll [7] in 1965, seemingly to avoid introduction of internal energy and use of the balance of energy in the purely mechanical development, replace the above assumptions with the assumption that the stress is determined by the deformation gradient history only to within a stress that does not do work in any motions satisfying the constraint. As shown by Carlson and Tortorelli [5], in the purely mechanical case for hyperelastic materials this starting assumption of Truesdell and Noll is satisfied when starting from the balance of energy. Obviously, Truesdell and Noll do not restrict their attention only to hyperelastic materials.

Rivlin [13] in 1966 starting from the Clausius–Duhem inequality constructed the form of the stress assuming the existence of multiple kinematical constraints. The procedure presented by Rivlin is similar to that presented in this article, but using the idea of Lagrange multipliers to remove the constraints on the velocity gradient.

Green et al. [15] in 1970 studied a non-holonomic constraint that is incompatible with the general thermoelastic constraint presented in this article. The constraint equation they studied was given as

$$\gamma_{ij}d_{ij} + \gamma_i\theta_{,i} = 0,$$

where d_{ij} is the velocity gradient, $\theta_{,i}$ is the temperature gradient, and γ_{ij} and γ_i are coefficients that are functions that do not depend on d_{ij} and $\theta_{,i}$. As a result of this particular selection, constraints such as the isothermally incompressible material constraint presented here are not covered by their study. The authors clearly discussed the possibility of the inclusion of a term of the form $\gamma\dot{\theta}$, but argue against it for two reasons. First, they exclude this term because this would add an indeterminate term into the entropy. Second, they exclude this term “since its physical significance is not clear” to them. These authors provide three methods to obtain the expressions for stress, entropy and heat flux. The first method they present is based on assuming an additive decomposition of each term (entropy, stress, and heat flux) into a constitutively prescribed part and a term resulting from the constraint, with the additional assumption that entropy production due to the term resulting from the constraint is non-negative. The second method is based on eliminating from the Clausius–Duhem inequality the one component of the velocity gradient which is associated with the reduction in the degrees of freedom resulting from the constraint, and then assuming certain resulting coefficients in the Clausius–Duhem inequality are constitutively prescribed. The third method starts with the assumption that each of the four quantities of free-energy, entropy, stress, and heat flux are given by constitutive expressions up to an additive term resulting from the constraint. The last two methods have elements which resemble the procedures presented here.

Gurtin and Podio-Guidugli [17] in 1973 started from an additive decomposition of stress, entropy, and heat flux into constitutively prescribed and reaction induced parts, arrive at their conclusions introducing a reaction set that is closed under scalar multiplication and assuming

that the reaction functional is maximal. For the thermodynamic case they show that they can derive the equation that is the starting point of the first method given by Green et al. [15].

Pipkin [24] in 1976 and Rostamian [25] in 1981 have studied internal constraints in linear elasticity.

Batra [18] in 1987 uses Hamilton's principle to study and derive the equations for internally constrained elastic materials considering both theories that use temperature and theories that use entropy as their independent variable.

Cohen and Wang [10] in 1987 and Podio-Guidugli and Vianello [26] in 1989 study issues related to material symmetry in constrained elastic materials.

Carlson and Tortorelli [5] in 1996 looked at hyperelastic materials and through geometric arguments arrive from the balance of energy to the expression for the decomposition of stress, without use of the Lagrange multiplier method. Their method does not require that the internal energy be extended off the constraint manifold.

Casey and Krishnaswamy [19] in 1998 studied internal constraints in thermoelastic materials following a procedure proposed by Rivlin that does not, a priori, assume the existence of an entropy function. By the a priori assumption that certain constitutive functions do not depend on temperature gradient, the procedure provides an entropy function based on Part I of the Second Law of Thermodynamics that asserts that the Clausius integral is path independent in strain-temperature space. In addition to discussing the non-uniqueness of the extension of the response functions off the constraint manifold and issues associated with this, they introduce the "intrinsic" response of the constrained material that results when a curvilinear coordinate is put on the constraint manifold, separating the response into a unique tangential part to the constraint manifold that describes the constitutively prescribed part and a normal part associated with the reaction to the constraint.

Saccomandi and Beatty [27] in 2002 consider universal relations and solutions for inextensible and incompressible isotropic elastic materials, also providing references to related work on universal relations for constrained materials.

10. SUMMARY, COMMENTS AND CONCLUSIONS

The underlying assumption of this article is that the material response must at all times and under all conditions satisfy the Clausius–Duhem inequality. Using this starting assumption, a procedure is developed to obtain the form of the response of thermoelastic materials in the presence of single and multiple thermomechanical constraints. The procedure is different from other developments in that the inability of the traditional variables to fully characterize the response is recognized in the constitutive relations by adding new arguments to make the constitutive equations complete. The Clausius–Duhem inequality then provides the explicit form of the indeterminate part of the response.

In the second part of this article it is pointed out that the separation of the part of the response that is constitutively prescribed from that which is in reaction to the constraint is not unique and that many different criteria may be used to accomplish this separation. Three methods for such separation are studied. Each method makes the two parts orthogonal, where the definition of orthogonality is different for each method.

In the third part of this paper examples are presented for both single and multiple constraints and it is shown how one can introduce isotropic and anisotropic thermal expansion into constraints that are normally considered only under isothermal conditions. A number of additional examples, particularly related to the Bell constraint, are included in the Appendix.

Of the examples, the one on zero temperature gradient normal to a material surface is different from all the others in that it imposes a constraint that includes the temperature gradient. This constraint introduces an indeterminate shear stress on the surface along the direction of the temperature gradient. Also, as a result of the constraint, the heat flux normal to the surface is no longer restricted by the Clausius–Duhem inequality and may take any quantity.

APPENDIX A. PROOF OF LEMMA

The proof of the lemma follows. To prove that $\mathcal{G}_{n+1}(\alpha_1, \dots, \alpha_m) \leq 0$, select all $\varpi_i = 0$. To show that $\mathcal{G}_1(\alpha_1, \dots, \alpha_m) = 0$, select $\varpi_i = 0$ for $i = 2, \dots, n$. Now start with the assumption that $\mathcal{G}_1(\alpha_1, \dots, \alpha_m) \neq 0$, say, for example, take $\mathcal{G}_1(\alpha_1, \dots, \alpha_m) > 0$. In this case let ϖ_1 be a positive number. Since $\mathcal{G}_1(\alpha_1, \dots, \alpha_m)$ and $\mathcal{G}_{n+1}(\alpha_1, \dots, \alpha_m)$ are bounded and independent of ϖ_1 , and $\mathcal{G}_1(\alpha_1, \dots, \alpha_m) > 0$, and since ϖ_1 can take any desired value, one can always increase ϖ_1 until $\mathcal{G}_1(\alpha_1, \dots, \alpha_m)\varpi_1$ becomes larger than the absolute value of $\mathcal{G}_{n+1}(\alpha_1, \dots, \alpha_m)$, so that $\mathcal{G}_1(\alpha_1, \dots, \alpha_m)\varpi_1 + \mathcal{G}_{n+1}(\alpha_1, \dots, \alpha_m) > 0$, which results in the violation of the initial assumption (9). In a similar way, one can show that $\mathcal{G}_1(\alpha_1, \dots, \alpha_m)$ cannot be less than zero, therefore leaving $\mathcal{G}_1(\alpha_1, \dots, \alpha_m) = 0$ as the only admissible possibility.

APPENDIX B. FURTHER EXAMPLES

Isothermally Bell constrained with isotropic thermal expansion: This constraint refers to Bell’s observation that $\text{tr}(\mathbf{V}) = 3$ during plastic flow of soft metals, where \mathbf{V} is the left symmetric factor in the polar decomposition of $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$. Here an isothermally Bell constrained material will be considered for which $\text{tr}(\bar{\mathbf{V}}) = 3$, where $\bar{\mathbf{V}}$ is the left symmetric factor in the polar decomposition of $\bar{\mathbf{F}}$ shown in Figure 4. For the case of isotropic thermal expansion one can write $\mathbf{U}^* = J^*(\theta)^{1/3}\mathbf{I}$, where the reference configuration is selected as any unloaded configuration at reference temperature. In view of $\mathbf{F} = J^{*1/3}\bar{\mathbf{F}}$, one can write this constraint as

$$\text{tr}(\mathbf{V}) = 3J^{*1/3}(\theta). \quad (202)$$

It should be pointed out that volume will not be constant during isothermal loading for a Bell constrained material as was shown by Beatty and Hayes [23]. For the constraint written as $f = \text{tr}(\mathbf{V}) - 3J^{*1/3}(\theta) = 0$, the partial derivatives⁷ are

$$\partial_{\mathbf{F}}(f) = \mathbf{R}, \quad \partial_{\theta}(f) = -J^{*-2/3} \frac{dJ^*}{d\theta}. \quad (203)$$

These result in the following expressions for Cauchy stress and entropy:

$$\mathbf{T} = \mathbf{T}_E + \varpi \mathbf{V}, \quad \eta = \eta_E + \frac{\varpi}{\rho} J^{*-2/3} \frac{dJ^*}{d\theta}. \quad (204)$$

The grouping of terms in the three methods presented result in the following expressions:

$$\begin{aligned} \mathbf{T}_E^{(1)T} &= \mathbf{T}_E^T - \frac{\text{tr}(\mathbf{T}_E \mathbf{V}^{-1}) + \rho \eta_E J^{*-2/3} \frac{dJ^*}{d\theta}}{3 + \left(J^{*-2/3} \frac{dJ^*}{d\theta} \right)^2} \mathbf{V}, \\ \mathbf{T}_R^{(1)T} &= \frac{\text{tr}(\mathbf{T} \mathbf{V}^{-1}) + \rho \eta J^{*-2/3} \frac{dJ^*}{d\theta}}{3 + \left(J^{*-2/3} \frac{dJ^*}{d\theta} \right)^2} \mathbf{V}, \end{aligned} \quad (205)$$

$$\begin{aligned} \eta_E^{(1)} &= \eta_E - \frac{\text{tr}(\mathbf{T}_E \mathbf{V}^{-1}) + \rho \eta_E J^{*-2/3} \frac{dJ^*}{d\theta}}{\rho \left[3 + \left(J^{*-2/3} \frac{dJ^*}{d\theta} \right)^2 \right]} J^{*-2/3} \frac{dJ^*}{d\theta}, \\ \eta_R^{(1)} &= \frac{\text{tr}(\mathbf{T} \mathbf{V}^{-1}) + \rho \eta J^{*-2/3} \frac{dJ^*}{d\theta}}{\rho \left[3 + \left(J^{*-2/3} \frac{dJ^*}{d\theta} \right)^2 \right]} J^{*-2/3} \frac{dJ^*}{d\theta}, \end{aligned} \quad (206)$$

$$\mathbf{T}_E^{(2)T} = \mathbf{T}_E^T - \frac{\text{tr}(\mathbf{T}_E \mathbf{V}^{-1})}{3} \mathbf{V}, \quad \mathbf{T}_R^{(2)T} = \frac{\text{tr}(\mathbf{T} \mathbf{V}^{-1})}{3} \mathbf{V}, \quad (207)$$

$$\eta_E^{(2)} = \eta_E - \frac{\text{tr}(\mathbf{T}_E \mathbf{V}^{-1})}{3\rho} J^{*-2/3} \frac{dJ^*}{d\theta}, \quad \eta_R^{(2)} = \frac{\text{tr}(\mathbf{T} \mathbf{V}^{-1})}{3\rho} J^{*-2/3} \frac{dJ^*}{d\theta}, \quad (208)$$

$$\mathbf{T}_E^{(3)T} = \mathbf{T}_E^T - \frac{\text{tr}(\mathbf{T}_E \mathbf{V})}{\text{tr}(\mathbf{B})} \mathbf{V}, \quad \mathbf{T}_R^{(3)T} = \frac{\text{tr}(\mathbf{T} \mathbf{V})}{\text{tr}(\mathbf{B})} \mathbf{V}, \quad (209)$$

$$\eta_E^{(3)} = \eta_E - \frac{\text{tr}(\mathbf{T}_E \mathbf{V})}{\rho \text{tr}(\mathbf{B})} J^{*-2/3} \frac{dJ^*}{d\theta}, \quad \eta_R^{(3)} = \frac{\text{tr}(\mathbf{T} \mathbf{V})}{\rho \text{tr}(\mathbf{B})} J^{*-2/3} \frac{dJ^*}{d\theta}. \quad (210)$$

Constraint on $\text{tr}(\mathbf{C})$: The constraint considered here is written as

$$\text{tr}(\mathbf{C}) = 3J^{*2/3}(\theta), \quad (211)$$

where $J^*(\theta)$ is the volume ratio representing the unloaded volumetric thermal expansion of the material assuming isotropic thermal expansion, with the reference configuration selected as any unloaded configuration at reference temperature. As for the case of the Bell constraint, the volume is not constant during loading under this constraint. For these conditions, this constraint is equivalent to $\text{tr}(\mathbf{C}) = 3$ for $\mathbf{C} = \mathbf{F}^T \mathbf{F}$ where \mathbf{F} is as described in Figure 4. For the constraint written as $f = \text{tr}(\mathbf{C}) - 3J^{*2/3}(\theta) = 0$, the partial derivatives are

$$\partial_{\mathbf{F}}(f) = 2\mathbf{F}, \quad \partial_{\theta}(f) = -2J^{*-1/3} \frac{dJ^*}{d\theta}. \quad (212)$$

These result in the following expressions for Cauchy stress and entropy:

$$\mathbf{T} = \mathbf{T}_E + 2\varpi \mathbf{B}, \quad \eta = \eta_E + \frac{2\varpi}{\rho} J^{*-1/3} \frac{dJ^*}{d\theta}. \quad (213)$$

The grouping of terms in the three methods presented result in the following expressions:

$$\begin{aligned} \mathbf{T}_E^{(1)T} &= \mathbf{T}_E^T - \frac{\text{tr}(\mathbf{T}_E) + \rho \eta_E J^{*-1/3} \frac{dJ^*}{d\theta}}{3J^{*2/3} + \left(J^{*-1/3} \frac{dJ^*}{d\theta} \right)^2} \mathbf{B}, \\ \mathbf{T}_R^{(1)T} &= \frac{\text{tr}(\mathbf{T}) + \rho \eta J^{*-1/3} \frac{dJ^*}{d\theta}}{3J^{*2/3} + \left(J^{*-1/3} \frac{dJ^*}{d\theta} \right)^2} \mathbf{B}, \end{aligned} \quad (214)$$

$$\begin{aligned} \eta_E^{(1)} &= \eta_E - \frac{\text{tr}(\mathbf{T}_E) - \rho \eta_E J^{*-1/3} \frac{dJ^*}{d\theta}}{\rho \left[3J^{*2/3} + \left(J^{*-1/3} \frac{dJ^*}{d\theta} \right)^2 \right]} J^{*-1/3} \frac{dJ^*}{d\theta}, \\ \eta_R^{(1)} &= \frac{\text{tr}(\mathbf{T}) + \rho \eta J^{*-1/3} \frac{dJ^*}{d\theta}}{\rho \left[3J^{*2/3} + \left(J^{*-1/3} \frac{dJ^*}{d\theta} \right)^2 \right]} J^{*-1/3} \frac{dJ^*}{d\theta}, \end{aligned} \quad (215)$$

$$\mathbf{T}_E^{(2)T} = \mathbf{T}_E^T - \frac{\text{tr}(\mathbf{T}_E)}{3J^{*2/3}} \mathbf{B}, \quad \mathbf{T}_R^{(2)T} = \frac{\text{tr}(\mathbf{T})}{3J^{*2/3}} \mathbf{B}, \quad (216)$$

$$\eta_E^{(2)} = \eta_E - \frac{\text{tr}(\mathbf{T}_E)}{3\rho J^{*2/3}} J^{*-1/3} \frac{dJ^*}{d\theta}, \quad \eta_R^{(2)} = \frac{\text{tr}(\mathbf{T})}{3\rho J^{*2/3}} J^{*-1/3} \frac{dJ^*}{d\theta}, \quad (217)$$

$$\mathbf{T}_E^{(3)T} = \mathbf{T}_E^T - \frac{\text{tr}(\mathbf{T}_E \mathbf{B})}{\text{tr}(\mathbf{B}^2)} \mathbf{B}, \quad \mathbf{T}_R^{(3)T} = \frac{\text{tr}(\mathbf{T} \mathbf{B})}{\text{tr}(\mathbf{B}^2)} \mathbf{B}, \quad (218)$$

$$\eta_E^{(3)} = \eta_E - \frac{\text{tr}(\mathbf{T}_E \mathbf{B})}{\rho \text{tr}(\mathbf{B}^2)} J^{*-1/3} \frac{dJ^*}{d\theta}, \quad \eta_R^{(3)} = \frac{\text{tr}(\mathbf{T} \mathbf{B})}{\rho \text{tr}(\mathbf{B}^2)} J^{*-1/3} \frac{dJ^*}{d\theta}. \quad (219)$$

Isothermally inextensible and Bell constrained: A material that is both isothermally inextensible along the direction $\hat{\mathbf{h}}_o$ in the reference configuration and isothermally Bell constrained is described by the two constraints

$$f_1 = \hat{\mathbf{h}}_o \circ (\mathbf{C}\hat{\mathbf{h}}_o) - \lambda^2(\theta) = 0, \quad (220)$$

$$f_2 = \text{tr}(\mathbf{V}) - 3J^{*1/3} = 0, \quad (221)$$

where J^* and λ are as described above for single constraints, and assuming isotropic thermal expansion associated with the Bell constraint. The associated partial derivatives are

$$\partial_{\mathbf{F}}(f_1) = 2(\mathbf{F}\hat{\mathbf{h}}_o) \otimes \hat{\mathbf{h}}_o, \quad \partial_{\theta}(f_1) = -2\lambda \frac{d\lambda}{d\theta}, \quad (222)$$

$$\partial_{\mathbf{F}}(f_2) = \mathbf{R}, \quad \partial_{\theta}(f_2) = -J^{*-2/3} \frac{dJ^*}{d\theta}. \quad (223)$$

This results in the expressions for Cauchy stress and entropy as

$$\mathbf{T}^T = \mathbf{T}_E^T + 2\varpi_1(\mathbf{F}\hat{\mathbf{h}}_o) \otimes (\mathbf{F}\hat{\mathbf{h}}_o) + \varpi_2 \mathbf{V}, \quad (224)$$

$$\eta = \eta_E + \frac{2\varpi_1}{\rho} \lambda \frac{d\lambda}{d\theta} + \frac{\varpi_2}{\rho} J^{*-2/3} \frac{dJ^*}{d\theta}. \quad (225)$$

The matrix $[K]$ with components $K_{ij} = [\partial_{\mathbf{F}}(f_i)\mathbf{F}^T] : [\partial_{\mathbf{F}}(f_j)\mathbf{F}^T]$ is, therefore, given by

$$[K] = \begin{bmatrix} 4\lambda^2 & 2\lambda^2 \hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}}) \\ 2\lambda^2 \hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}}) & \text{tr}(\mathbf{B}) \end{bmatrix}. \quad (226)$$

The inverse of this matrix is given by

$$[K]^{-1} = \frac{1}{4\lambda^4 \{\text{tr}(\mathbf{B}) - [\hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}})]^2\}} \begin{bmatrix} \text{tr}(\mathbf{B}) & -2\lambda^2 \hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}}) \\ -2\lambda^2 \hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}}) & 4\lambda^2 \end{bmatrix}, \quad (227)$$

which results in the expressions

$$\begin{aligned} \mathbf{T}_E^{(3)T} &= \mathbf{T}_E^T - \frac{\hat{\mathbf{h}} \circ (\mathbf{T}_E \hat{\mathbf{h}}) \text{tr}(\mathbf{B}) - \hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}}) \text{tr}(\mathbf{T}_E \mathbf{V})}{\text{tr}(\mathbf{B}) - [\hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}})]^2} \hat{\mathbf{h}} \otimes \hat{\mathbf{h}} \\ &+ \frac{\hat{\mathbf{h}} \circ (\mathbf{T}_E \hat{\mathbf{h}}) \hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}}) - \text{tr}(\mathbf{T}_E \mathbf{V})}{\text{tr}(\mathbf{B}) - [\hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}})]^2} \mathbf{V}, \end{aligned} \quad (228)$$

$$\begin{aligned} \mathbf{T}_R^{(3)T} &= \frac{\hat{\mathbf{h}} \circ (\mathbf{T}\hat{\mathbf{h}}) \text{tr}(\mathbf{B}) - \hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}}) \text{tr}(\mathbf{T}\mathbf{V})}{\text{tr}(\mathbf{B}) - [\hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}})]^2} \hat{\mathbf{h}} \otimes \hat{\mathbf{h}} \\ &- \frac{\hat{\mathbf{h}} \circ (\mathbf{T}\hat{\mathbf{h}}) \hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}}) - \text{tr}(\mathbf{T}\mathbf{V})}{\text{tr}(\mathbf{B}) - [\hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}})]^2} \mathbf{V}, \end{aligned} \quad (229)$$

$$\begin{aligned} \eta_E^{(3)} &= \eta_E - \frac{\hat{\mathbf{h}} \circ (\mathbf{T}_E \hat{\mathbf{h}}) \text{tr}(\mathbf{B}) - \hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}}) \text{tr}(\mathbf{T}_E \mathbf{V})}{\text{tr}(\mathbf{B}) - [\hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}})]^2} \frac{\lambda}{\rho} \frac{d\lambda}{d\theta} \\ &+ \frac{\hat{\mathbf{h}} \circ (\mathbf{T}_E \hat{\mathbf{h}}) \hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}}) - \text{tr}(\mathbf{T}_E \mathbf{V})}{\text{tr}(\mathbf{B}) - [\hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}})]^2} \frac{J^{*-2/3}}{\rho} \frac{dJ^*}{d\theta}, \end{aligned} \quad (230)$$

$$\begin{aligned} \eta_R^{(3)} &= \frac{\hat{\mathbf{h}} \circ (\mathbf{T}\hat{\mathbf{h}}) \operatorname{tr}(\mathbf{B}) - \hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}}) \operatorname{tr}(\mathbf{T}\mathbf{V})}{\operatorname{tr}(\mathbf{B}) - [\hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}})]^2} \frac{\lambda}{\rho} \frac{d\lambda}{d\theta} \\ &- \frac{\hat{\mathbf{h}} \circ (\mathbf{T}\hat{\mathbf{h}}) \hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}}) - \operatorname{tr}(\mathbf{T}\mathbf{V})}{\operatorname{tr}(\mathbf{B}) - [\hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}})]^2} \frac{J^{*-2/3}}{\rho} \frac{dJ^*}{d\theta}. \end{aligned} \quad (231)$$

Isothermally Bell constrained and inextensible along two directions: A material that is isothermally inextensible along the two directions $\hat{\mathbf{h}}_o$ and $\hat{\mathbf{m}}_o$ in the reference configuration and is also Bell constrained is described by the following constraints:

$$f_1 = \hat{\mathbf{h}}_o \circ (\mathbf{C}\hat{\mathbf{h}}_o) - \lambda_1^2(\theta) = 0, \quad (232)$$

$$f_2 = \hat{\mathbf{m}}_o \circ (\mathbf{C}\hat{\mathbf{m}}_o) - \lambda_2^2(\theta) = 0, \quad (233)$$

$$f_3 = \operatorname{tr}(\mathbf{V}) - 3J^{*1/3}(\theta) = 0, \quad (234)$$

where λ_1 and λ_2 are the stretches associated with thermal expansions along the respective inextensibility directions and J^* is the volume ratio associated with thermal expansion, and assuming isotropic thermal expansion associated with the Bell constraint. The associated partial derivatives are

$$\partial_{\mathbf{F}}(f_1) = 2(\mathbf{F}\hat{\mathbf{h}}_o) \otimes \hat{\mathbf{h}}_o, \quad \partial_{\theta}(f_1) = -2\lambda_1 \frac{d\lambda_1}{d\theta}, \quad (235)$$

$$\partial_{\mathbf{F}}(f_2) = 2(\mathbf{F}\hat{\mathbf{m}}_o) \otimes \hat{\mathbf{m}}_o, \quad \partial_{\theta}(f_2) = -2\lambda_2 \frac{d\lambda_2}{d\theta}, \quad (236)$$

$$\partial_{\mathbf{F}}(f_3) = \mathbf{R}, \quad \partial_{\theta}(f_3) = -J^{*-2/3} \frac{dJ^*}{d\theta}. \quad (237)$$

This results in the expressions for Cauchy stress and entropy as

$$\mathbf{T}^T = \mathbf{T}_E^T + 2\varpi_1(\mathbf{F}\hat{\mathbf{h}}_o) \otimes (\mathbf{F}\hat{\mathbf{h}}_o) + 2\varpi_2(\mathbf{F}\hat{\mathbf{m}}_o) \otimes (\mathbf{F}\hat{\mathbf{m}}_o) + \varpi_3\mathbf{V}, \quad (238)$$

$$\eta = \eta_E + \frac{2\varpi_1}{\rho} \lambda_1 \frac{d\lambda_1}{d\theta} + \frac{2\varpi_2}{\rho} \lambda_2 \frac{d\lambda_2}{d\theta} + \frac{\varpi_3}{\rho} J^{*-2/3} \frac{dJ^*}{d\theta}. \quad (239)$$

The matrix $[K]$ with components $K_{ij} = [\partial_{\mathbf{F}}(f_i)\mathbf{F}^T] : [\partial_{\mathbf{F}}(f_j)\mathbf{F}^T]$ is, therefore, given by

$$[K] = 4 \begin{bmatrix} \lambda_1^4 & \lambda_1^2 \lambda_2^2 (\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2 & 2\lambda_1^2 \hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}}) \\ \lambda_1^2 \lambda_2^2 (\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2 & \lambda_2^4 & 2\lambda_2^2 \hat{\mathbf{m}} \circ (\mathbf{V}\hat{\mathbf{m}}) \\ 2\lambda_1^2 \hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}}) & 2\lambda_2^2 \hat{\mathbf{m}} \circ (\mathbf{V}\hat{\mathbf{m}}) & \operatorname{tr}(\mathbf{B}) \end{bmatrix}, \quad (240)$$

where $\hat{\mathbf{h}} = \mathbf{F}\hat{\mathbf{h}}_o/\lambda_1$ and $\hat{\mathbf{m}} = \mathbf{F}\hat{\mathbf{m}}_o/\lambda_2$. The inverse of this matrix is given by

$$[K]^{-1} = \frac{1}{4\lambda_1^4\lambda_2^4D} \times \left[\begin{array}{cc} -\lambda_2^4\{\text{tr}(\mathbf{B}) - [\hat{\mathbf{m}} \circ (\mathbf{V}\hat{\mathbf{m}})]^2\} & \\ \lambda_1^2\lambda_2^2[(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2\text{tr}(\mathbf{B}) - \hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}})\hat{\mathbf{m}} \circ (\mathbf{V}\hat{\mathbf{m}})] & \\ -2\lambda_1^2\lambda_2^4[(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2\hat{\mathbf{m}} \circ (\mathbf{V}\hat{\mathbf{m}}) - \hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}})] & \\ \lambda_1^2\lambda_2^2[(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2\text{tr}(\mathbf{B}) - \hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}})\hat{\mathbf{m}} \circ (\mathbf{V}\hat{\mathbf{m}})] & -2\lambda_1^2\lambda_2^4[(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2\hat{\mathbf{m}} \circ (\mathbf{V}\hat{\mathbf{m}}) - \hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}})] \\ -\lambda_1^4\{\text{tr}(\mathbf{B}) - [\hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}})]^2\} & -2\lambda_1^4\lambda_2^2[(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2\hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}}) - \hat{\mathbf{m}} \circ (\mathbf{V}\hat{\mathbf{m}})] \\ -2\lambda_1^4\lambda_2^2[(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2\hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}}) - \hat{\mathbf{m}} \circ (\mathbf{V}\hat{\mathbf{m}})] & 4\lambda_1^4\lambda_2^4[(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^4 - 1] \end{array} \right], \quad (241)$$

where

$$D = \{[\hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}})]^2 + [\hat{\mathbf{m}} \circ (\mathbf{V}\hat{\mathbf{m}})]^2 - 2(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2\hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}})\hat{\mathbf{m}} \circ (\mathbf{V}\hat{\mathbf{m}}) + [(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^4 - 1]\text{tr}(\mathbf{B})\}. \quad (242)$$

From $[K]^{-1}$ one can calculate $\bar{\varpi}_i^{(3)}$ and $\varpi_i^{*(3)}$ as follows

$$\begin{aligned} \bar{\varpi}_1^{(3)} &= -\frac{1}{2\lambda_1^2D} \{ \hat{\mathbf{h}} \circ (\mathbf{T}_E\hat{\mathbf{h}}) [(\hat{\mathbf{m}} \circ (\mathbf{V}\hat{\mathbf{m}})]^2 - \text{tr}(\mathbf{B})) + \hat{\mathbf{m}} \circ (\mathbf{T}_E\hat{\mathbf{m}}) [(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2\text{tr}(\mathbf{B}) \\ &\quad - \hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}})\hat{\mathbf{m}} \circ (\mathbf{V}\hat{\mathbf{m}})] + \text{tr}(\mathbf{T}_E\mathbf{V}) [\hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}}) - (\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2\hat{\mathbf{m}} \circ (\mathbf{V}\hat{\mathbf{m}})] \}, \end{aligned} \quad (243)$$

$$\begin{aligned} \bar{\varpi}_2^{(3)} &= -\frac{1}{2\lambda_2^2D} \{ \hat{\mathbf{h}} \circ (\mathbf{T}_E\hat{\mathbf{h}}) [(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2\text{tr}(\mathbf{B}) - \hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}})\hat{\mathbf{m}} \circ (\mathbf{V}\hat{\mathbf{m}})] + \hat{\mathbf{m}} \circ (\mathbf{T}_E\hat{\mathbf{m}}) \\ &\quad \times ((\hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}})]^2 - \text{tr}(\mathbf{B})) + \text{tr}(\mathbf{T}_E\mathbf{V}) [\hat{\mathbf{m}} \circ (\mathbf{V}\hat{\mathbf{m}}) - (\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2\hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}})] \}, \end{aligned} \quad (244)$$

$$\begin{aligned} \bar{\varpi}_3^{(3)} &= -\frac{1}{D} \{ \hat{\mathbf{h}} \circ (\mathbf{T}_E\hat{\mathbf{h}}) [\hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}}) - (\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2\hat{\mathbf{m}} \circ (\mathbf{V}\hat{\mathbf{m}})] + \hat{\mathbf{m}} \circ (\mathbf{T}_E\hat{\mathbf{m}}) [\hat{\mathbf{m}} \circ (\mathbf{V}\hat{\mathbf{m}}) \\ &\quad - (\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^2\hat{\mathbf{h}} \circ (\mathbf{V}\hat{\mathbf{h}})] + \text{tr}(\mathbf{T}_E\mathbf{V}) [(\hat{\mathbf{h}} \circ \hat{\mathbf{m}})^4 - 1] \}. \end{aligned} \quad (245)$$

The expressions for $\varpi_i^{*(3)}$ are similar and can be obtained by removing the “-” sign preceding each expression and replacing \mathbf{T}_E with \mathbf{T} . The expression for stress can now be written as

$$\mathbf{T}_E^{(3)T} = \mathbf{T}_E^T + 2\lambda_1^2\bar{\varpi}_1^{(3)}\hat{\mathbf{h}} \otimes \hat{\mathbf{h}} + 2\lambda_2^2\bar{\varpi}_3^{(3)}\hat{\mathbf{m}} \otimes \hat{\mathbf{m}} + \bar{\varpi}_3^{(3)}\mathbf{V} \quad (246)$$

$$\mathbf{T}_R^{(3)T} = 2\lambda_1^2\varpi_1^{*(3)}\hat{\mathbf{h}} \otimes \hat{\mathbf{h}} + 2\lambda_2^2\varpi_3^{*(3)}\hat{\mathbf{m}} \otimes \hat{\mathbf{m}} + \varpi_3^{*(3)}\mathbf{V}, \quad (247)$$

$$\eta_E^{(3)} = \eta_E + \frac{2\bar{\varpi}_1^{(3)}\lambda_1}{\rho} \frac{d\lambda_1}{d\theta} + \frac{2\bar{\varpi}_2^{(3)}\lambda_2}{\rho} \frac{d\lambda_2}{d\theta} + \frac{\bar{\varpi}_3^{(3)}J^{*-2/3}}{\rho} \frac{dJ^*}{d\theta}, \quad (248)$$

$$\eta_R^{(3)} = \frac{2\varpi_1^{*(3)}\lambda_1}{\rho} \frac{d\lambda_1}{d\theta} + \frac{2\varpi_2^{*(3)}\lambda_2}{\rho} \frac{d\lambda_2}{d\theta} + \frac{\varpi_3^{*(3)}J^{*-2/3}}{\rho} \frac{dJ^*}{d\theta}. \quad (249)$$

NOTES

1. The selection of p as a scalar is not essential to the development. Without any change in the outcomes, p can be taken as a tensor of any rank. As will be seen later in the development, the imposition of one scalar valued constraint function results in the introduction of a single scalar function of p in the

expressions for Cauchy stress and entropy. That is, irrespective of the form of p , its influence is felt by the response through a single scalar.

2. This assumption is similar to the introduction of a one-parameter family of unconstrained materials, as proposed in [19].
3. For the purposes of this presentation one can assume that ψ^\dagger is a general function of \mathbf{X} and \mathcal{L} , even though as a result of the constraint ψ can only realize values of ψ^\dagger evaluated at points $(\mathbf{X}, \mathcal{L})$ consistent with the constraint condition $f(\mathbf{X}, \mathcal{L}) = 0$. That is, for each material point \mathbf{X} , even though ψ^\dagger is defined over the 13-dimensional space of \mathcal{L} , ψ can only take values of ψ^\dagger evaluated over the 12-dimensional constraint manifold defined by $f(\mathbf{X}, \mathcal{L}) = 0$.
If it is desired to assume that ψ^\dagger is only defined over the 12-dimensional constraint manifold, then for the derivatives to be meaningful as stated one has to extend this function off the constraint manifold. Such extensions are not unique. Any function $\psi^{\dagger\dagger}$ with the following properties can be used in place of ψ^\dagger in the development to make the derivatives stated meaningful. Take any function $\psi^{\dagger\dagger}$ defined over the 13-dimensional space of \mathcal{L} with the property that $\psi^{\dagger\dagger}$ have the same value as ψ^\dagger at every point on the constraint manifold, and that $\psi^{\dagger\dagger}$ be differentiable with respect to \mathcal{L} at each point on the constraint manifold. These conditions force $\dot{\psi}^{\dagger\dagger} = \dot{\psi}^\dagger$ at each point on the constraint manifold when restricting attention to motions on the constraint manifold. Note 5 addresses other issues that might arise with such non-unique extensions.
4. The terms “active stress” and “reactive stress” are also used for \mathbf{T}_E and \mathbf{T}_R , respectively.
5. In reference to the second part of Note 3, the non-uniqueness of the extension of the function for free-energy off the constraint manifold will not result in a non-uniqueness in the grouping for Method 1, but might in Method 2 and 3. For Method 1, the method separates $\partial_{\mathcal{L}}(\psi)$ into a part that is tangent to the constraint manifold and a part that is normal to it and then removes the non-unique normal part leaving only the unique tangential part to be used in the construction of the extra stress and extra entropy. As such, Method 1 is similar to the “intrinsic response” introduced by Casey and Krishnaswamy [19].
6. This relation directly follows from taking the derivative of the relation $\mathbf{U}^*\mathbf{U}^{*-1} = \mathbf{I}$.
7. The relation $\partial_{\mathbf{F}}(f) = \mathbf{R}$ can be shown as follows. Since $\mathbf{V}\mathbf{V}^{-1} = \mathbf{I}$ it follows after taking its derivative that

$$\frac{d\mathbf{V}^{-1}}{dt} = -\mathbf{V}^{-1}\dot{\mathbf{V}}\mathbf{V}^{-1}. \quad (250)$$

Since $\mathbf{V} = \mathbf{V}^{-1}\mathbf{B}$, it follows that

$$\dot{\mathbf{V}} = \frac{d\mathbf{V}^{-1}}{dt}\mathbf{B} + \mathbf{V}^{-1}\dot{\mathbf{B}} = -\mathbf{V}^{-1}\dot{\mathbf{V}}\mathbf{V}^{-1}\mathbf{B} + \mathbf{V}^{-1}\dot{\mathbf{B}}. \quad (251)$$

Therefore,

$$\text{tr}(\dot{\mathbf{V}}) = -\text{tr}(\mathbf{V}^{-1}\dot{\mathbf{V}}\mathbf{V}^{-1}\mathbf{B}) + \text{tr}(\mathbf{V}^{-1}\dot{\mathbf{B}}). \quad (252)$$

Since $\text{tr}(\mathbf{V}^{-1}\dot{\mathbf{V}}\mathbf{V}^{-1}\mathbf{B}) = \text{tr}(\dot{\mathbf{V}})$, it follows that

$$2\text{tr}(\dot{\mathbf{V}}) = \text{tr}(\mathbf{V}^{-1}\dot{\mathbf{B}}). \quad (253)$$

Using $\mathbf{B} = \mathbf{F}\mathbf{F}^T$, taking its time derivative to get $\dot{\mathbf{B}} = \dot{\mathbf{F}}\mathbf{F}^T + \mathbf{F}\dot{\mathbf{F}}^T$ and substituting back into the previous expression yields $\text{tr}(\dot{\mathbf{V}}) = \text{tr}(\mathbf{R}^T\dot{\mathbf{F}})$ that can be written as

$$\text{tr}(\dot{\mathbf{V}}) = \mathbf{R} : \dot{\mathbf{F}}. \quad (254)$$

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