Fundamentals of Finite Elements

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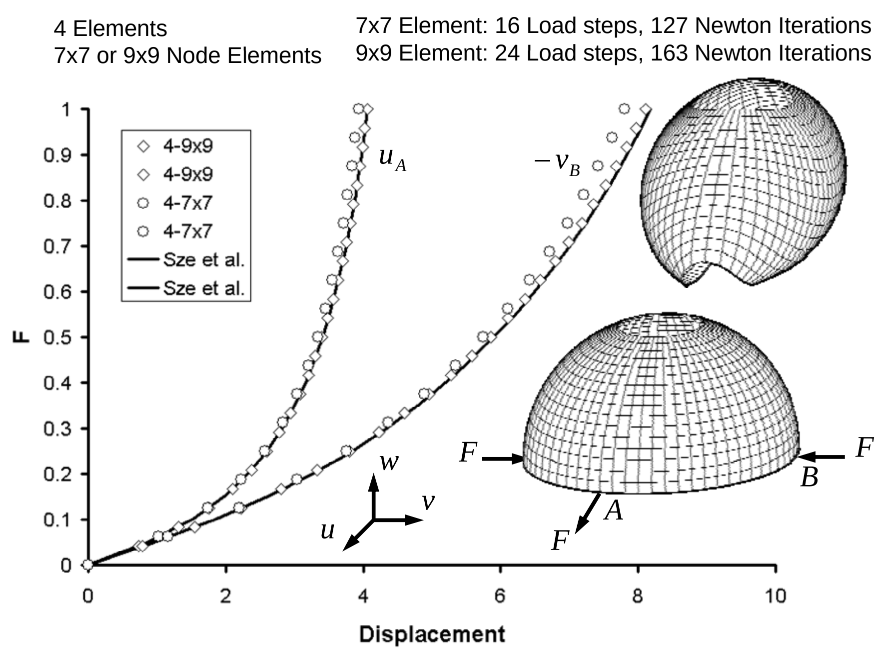
Course description

Objective:

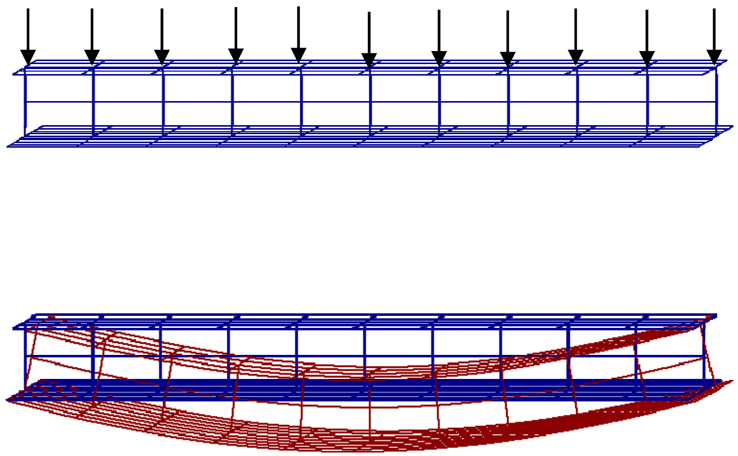
- Solution of partial differential equations of engineering importance using FEM.
- Emphasizing the similarity of the development.
- Development of common mathematical and numerical tools.
- Using consistent and robust methods.
- Developing an object oriented programming structure for FEM.
- Addressing some advanced topics.



Topics

Course

- Object oriented programming in FEM
- Weighted Residual Methods
- Linear Problems (scalar and vector fields, multi physics)
    - Heat transfer, elasticity, and thermoelasticity
- Constraints
- Non-Linear Problems
    - Nonlinear elasticity (large deformation)
- Initial-Value Problems
    - Transient thermal analysis
    - Vibration of beams
- Bars, Plates and Shells
    - Kinematically defined continua
- Numerical Implementation



Shells

Peanut

Sunflower

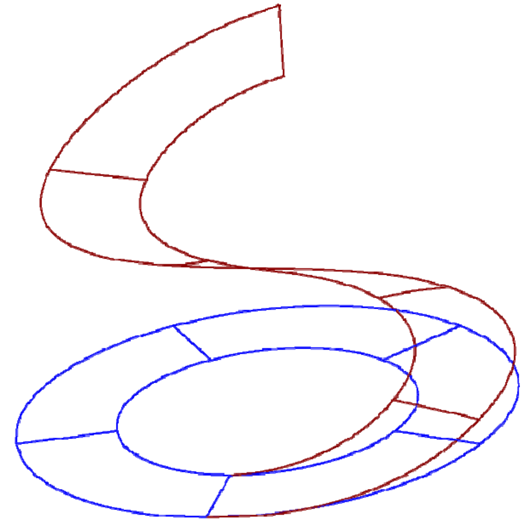
Eggs

Solanum cinereum germinating

Coconut

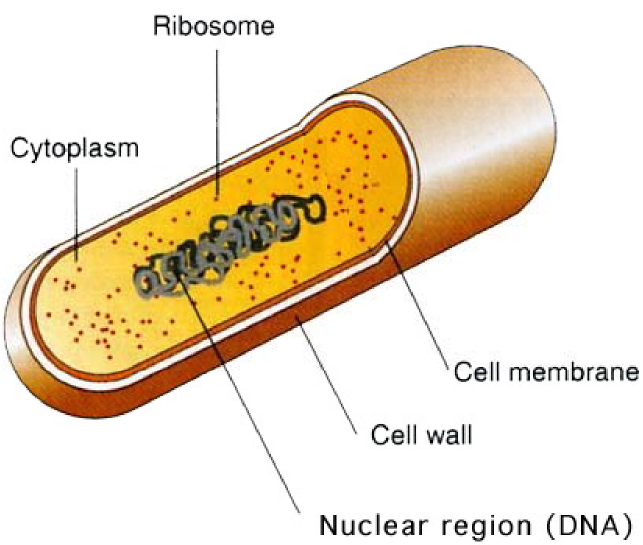
Shells

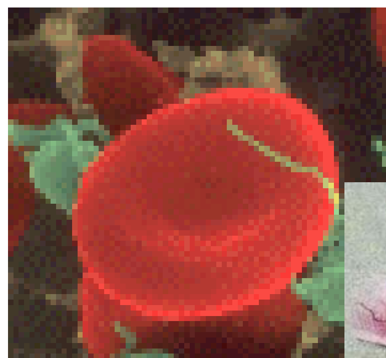
Apple



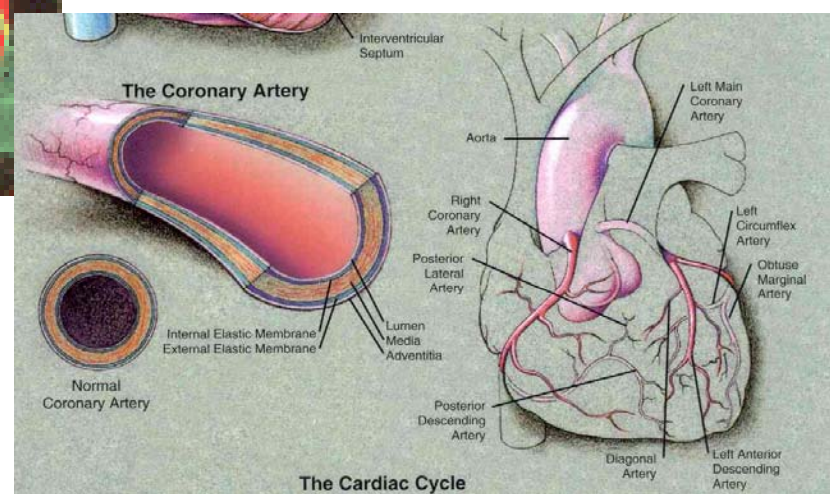
Red blood cell

Cell structure: Procaryote



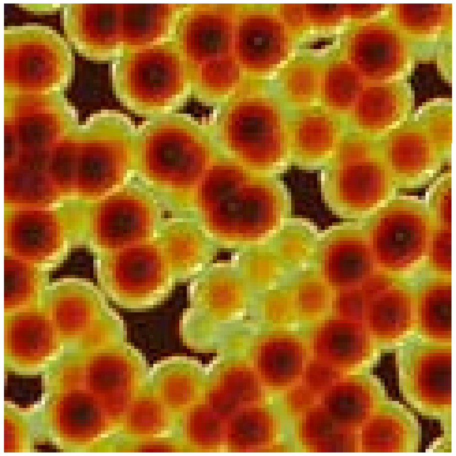


Coronary artery

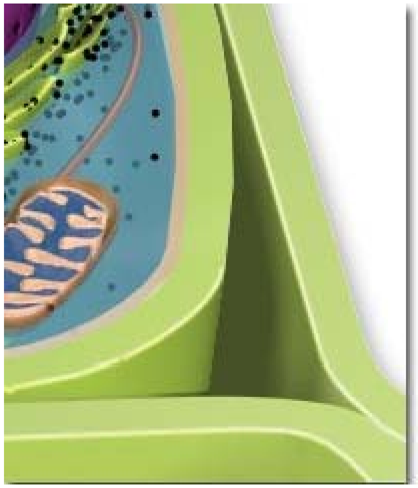


Shells

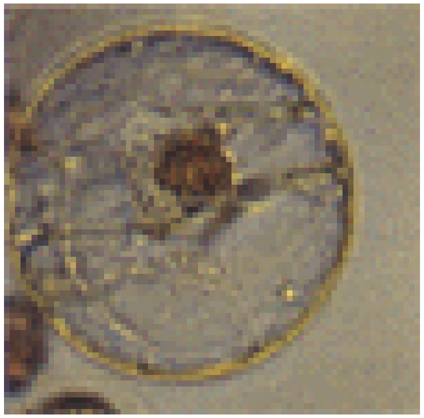
Plant cells



Plant cell wall

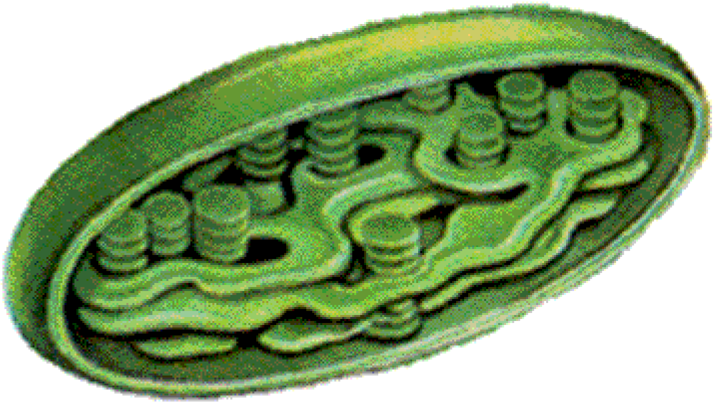


Poplar leaf cell

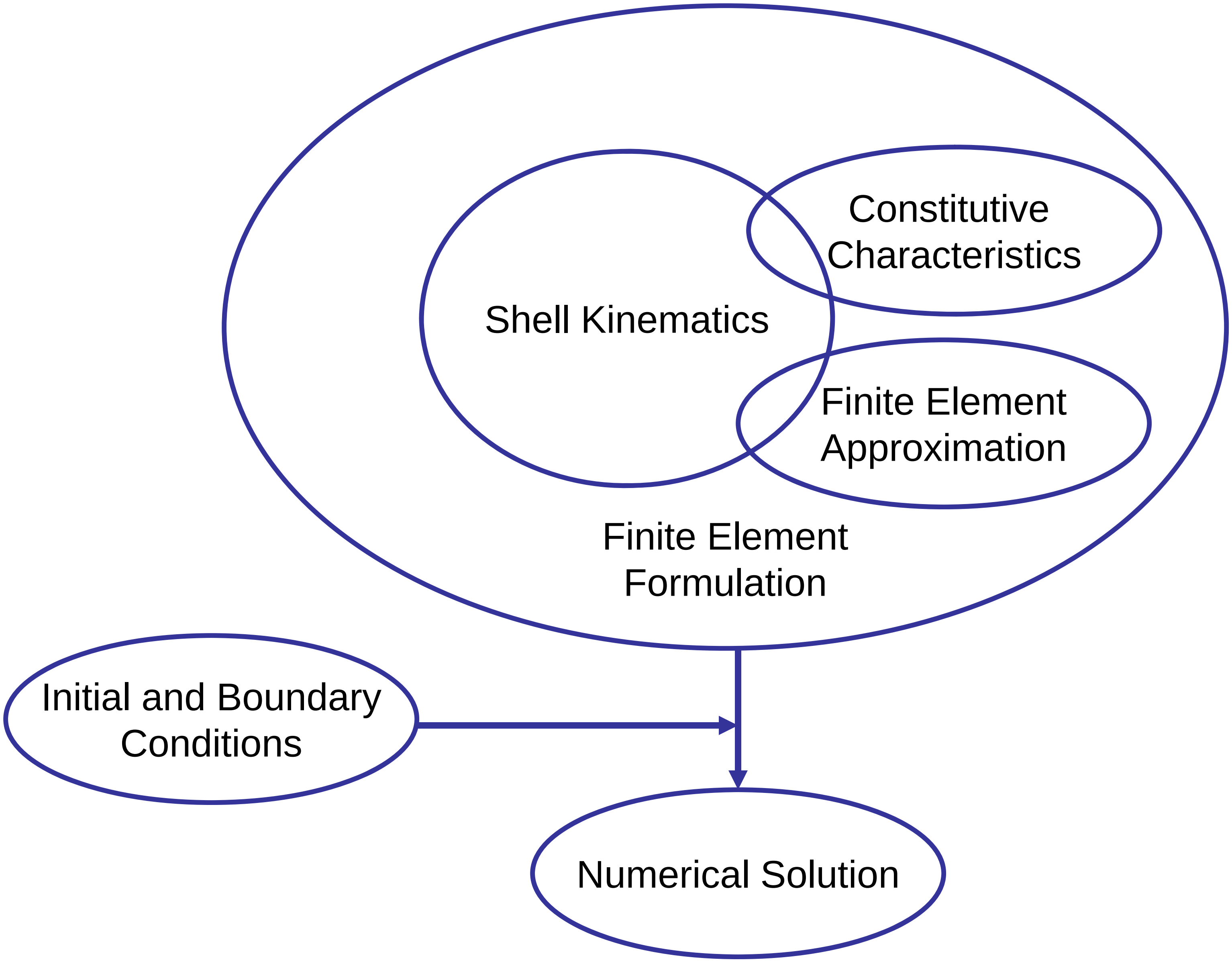


Tallow tree

Chloroplast: site of photosynthesis in plant cells



Typical FEM Formulation



Finite Element Formulation

Shell Kinematics

Finite Element Approximation

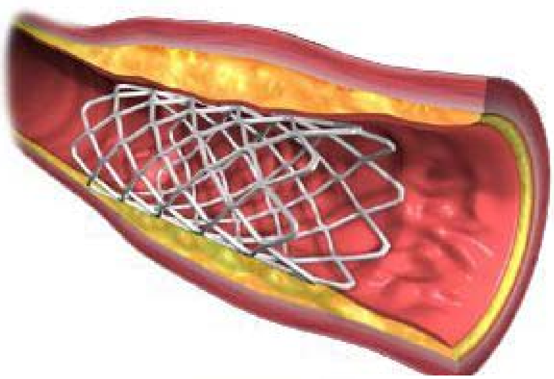
Variational Formulation

FEM Formulation

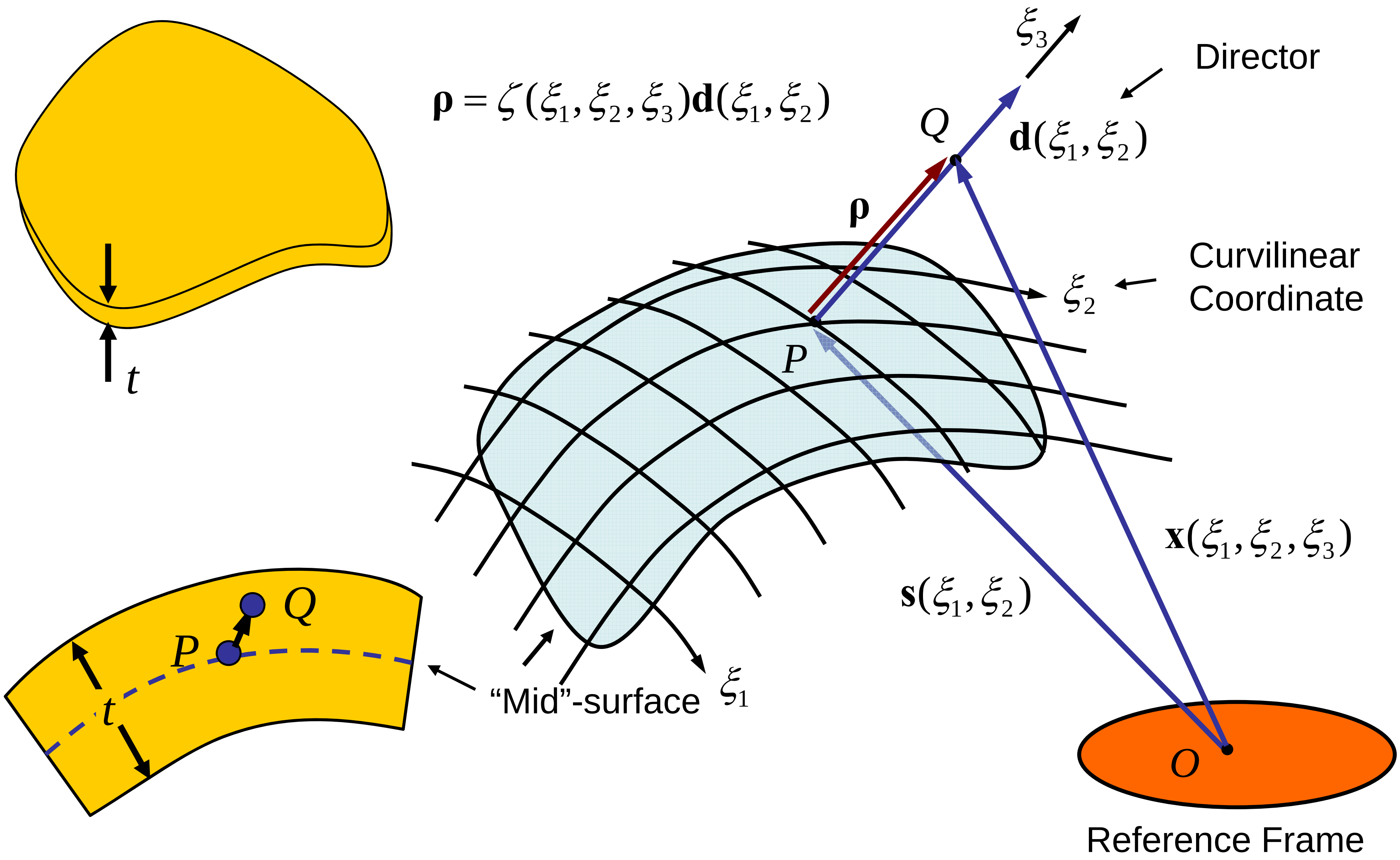
Constitutive Characteristics

Initial and Boundary Conditions

Numerical Solution



Shell Kinematics



Shell Kinematics

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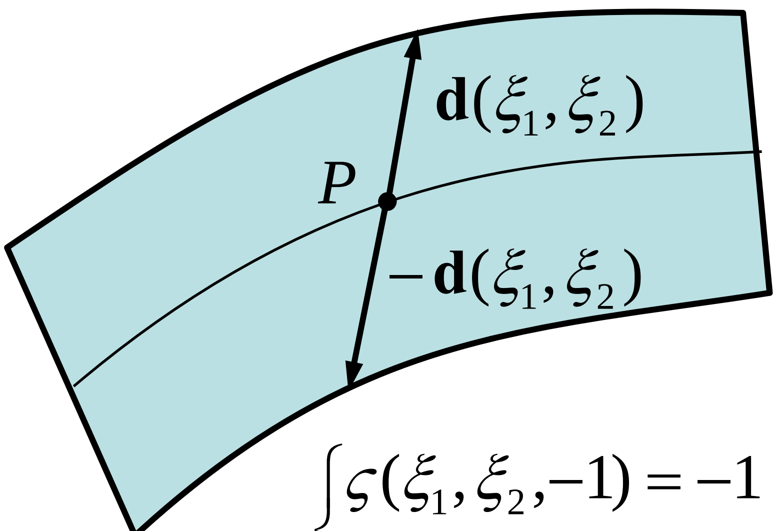
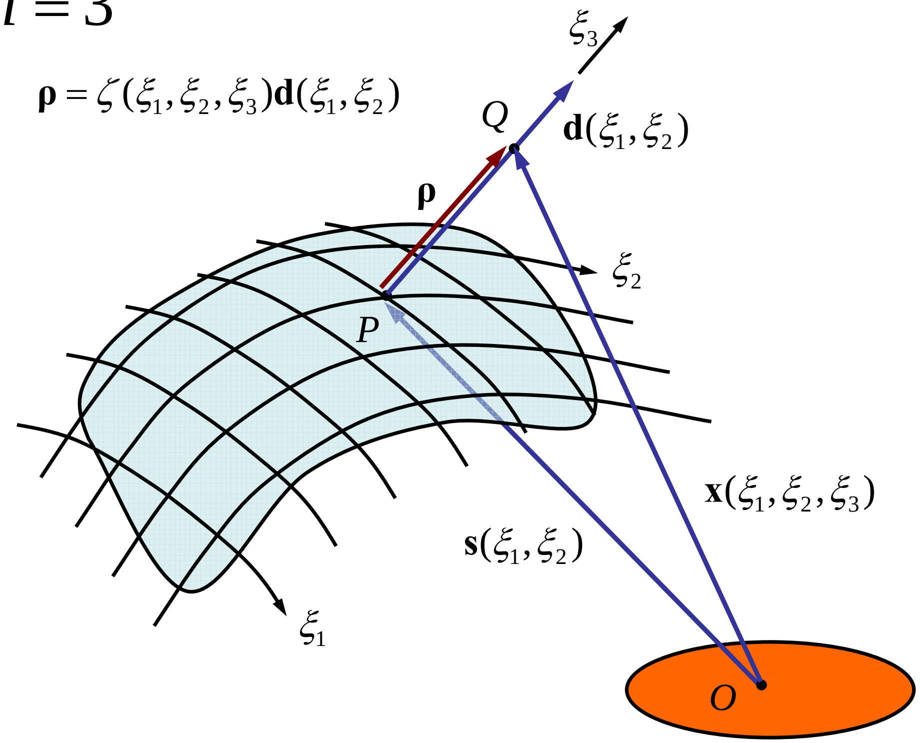
Reissner/Mindlin Shell

$$\mathbf{x}(\xi_1,\xi_2,\xi_3)=\mathbf{s}(\xi_1,\xi_2)+\varsigma(\xi_1,\xi_2,\xi_3)\mathbf{d}(\xi_1,\xi_2) \qquad g_{ij}=\mathbf{g}_i\circ\mathbf{g}_j$$

$$\mathbf{g}_i=\frac{\partial\mathbf{x}}{\partial\xi_i}=\begin{cases}\dfrac{\partial\mathbf{s}}{\partial\xi_i}+\dfrac{\partial\varsigma}{\partial\xi_i}\mathbf{d}+\varsigma\dfrac{\partial\mathbf{d}}{\partial\xi_i} & i=1,2\\[2ex] \dfrac{\partial\varsigma}{\partial\xi_3}\mathbf{d} & i=3\end{cases}$$

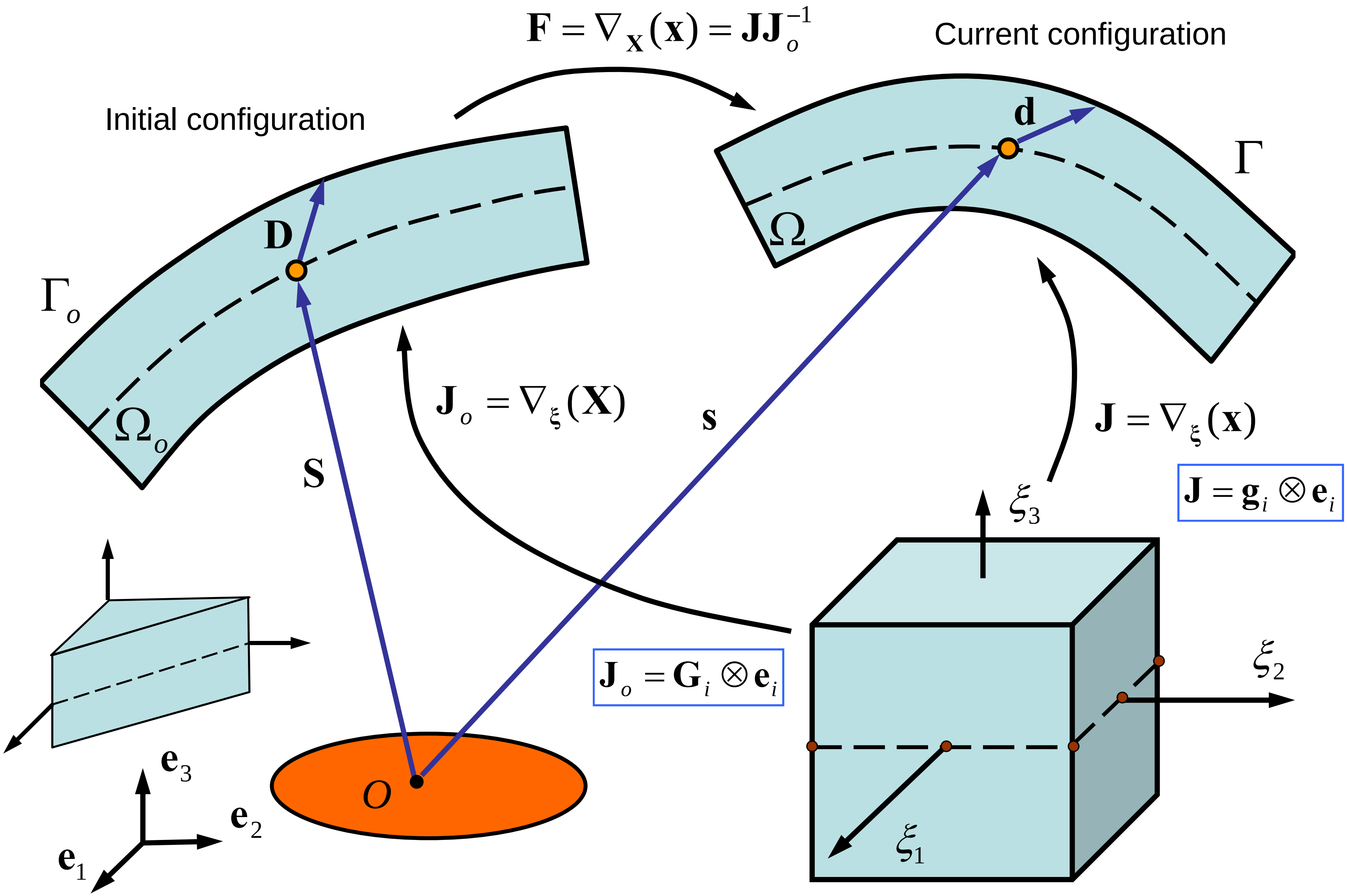


$$\begin{cases}\varsigma(\xi_1,\xi_2,-1)=-1\\ \varsigma(\xi_1,\xi_2,1)=1\end{cases}$$



Shell Kinematics

Making the mappings into material mappings:



Shell Kinematics

Deformation Gradient: $\mathbf{F} = \mathbf{J}\mathbf{J}_o^{-1}$

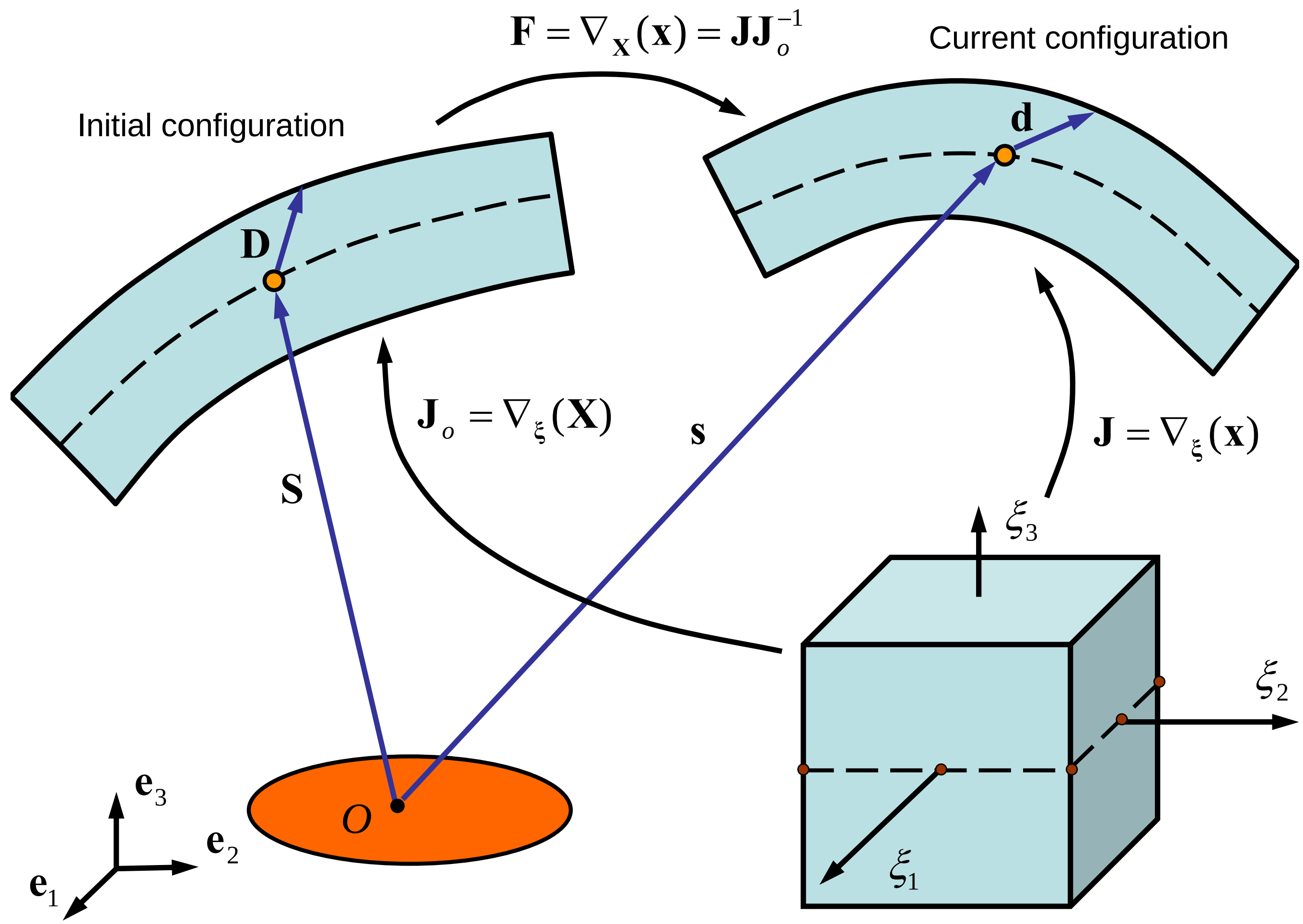
Displacement Gradient: $\mathbf{H} = \mathbf{F} - \mathbf{I}$

Strains:

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\mathbf{H} + \mathbf{H}^T)$$

$$\mathbf{G}_R = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$$

$$\mathbf{G}_L = \frac{1}{2}(\mathbf{F}\mathbf{F}^T - \mathbf{I})$$

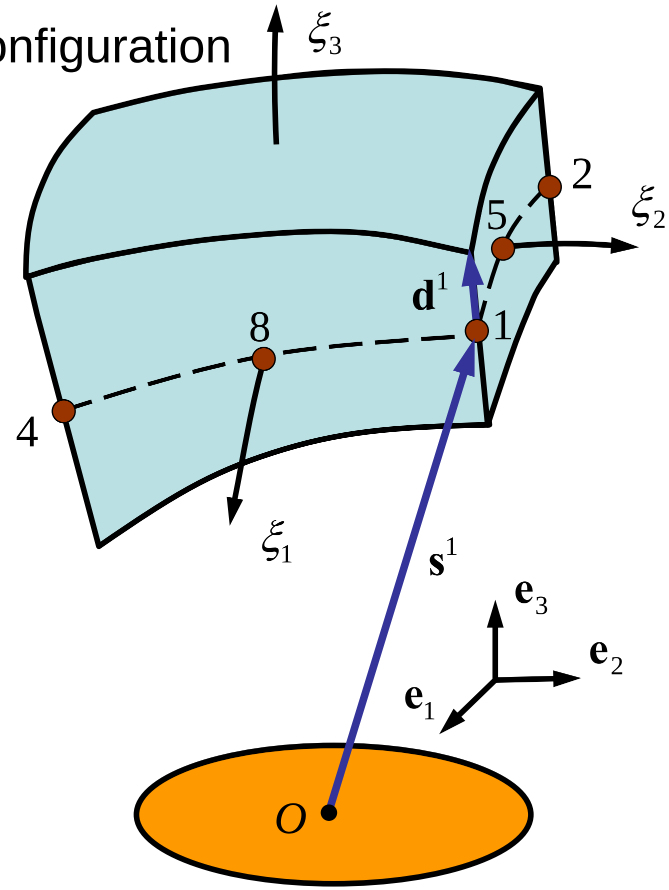


Isoparametric shell

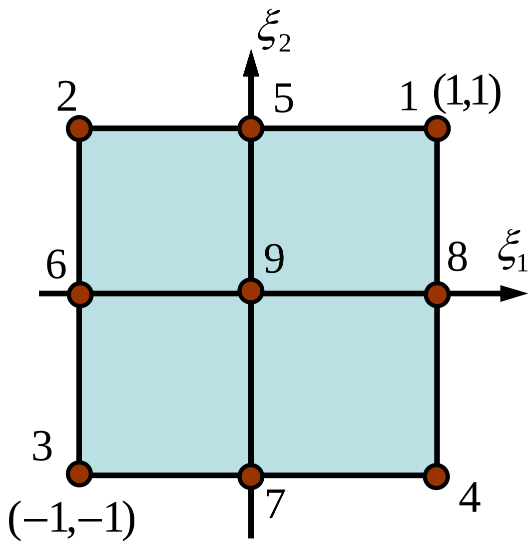
$$\mathbf{x}(\xi_1,\xi_2,\xi_3)=\mathbf{s}(\xi_1,\xi_2)+\varsigma(\xi_1,\xi_2,\xi_3)\mathbf{d}(\xi_1,\xi_2)$$

$$\boxed{\begin{aligned}
\mathbf{s}(\xi_1,\xi_2) &= \sum_{p=1}^{nnpe}\mathbf{s}^p N_p(\xi_1,\xi_2)\\
\mathbf{d}(\xi_1,\xi_2) &= \sum_{p=1}^{nnpe}\mathbf{d}^p N_p(\xi_1,\xi_2)\\
\varsigma(\xi_1,\xi_2,\xi_3) &= \sum_{p=1}^{nnpe}\varsigma^p(\xi_3) N_p(\xi_1,\xi_2)
\end{aligned}}$$

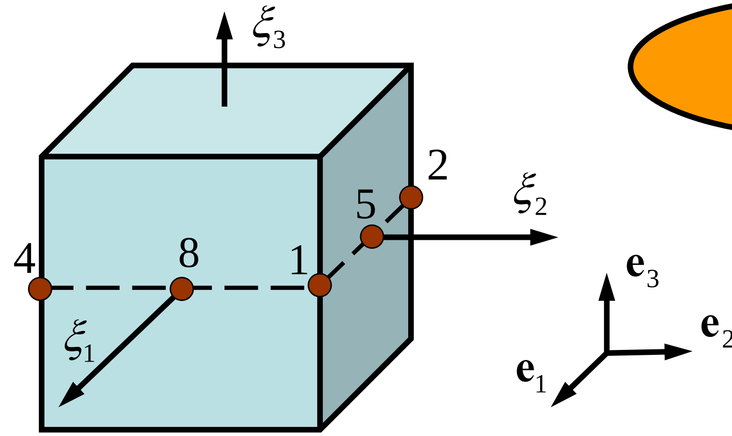
Current Configuration



Isoparametric 2-D



Extruded 3-D

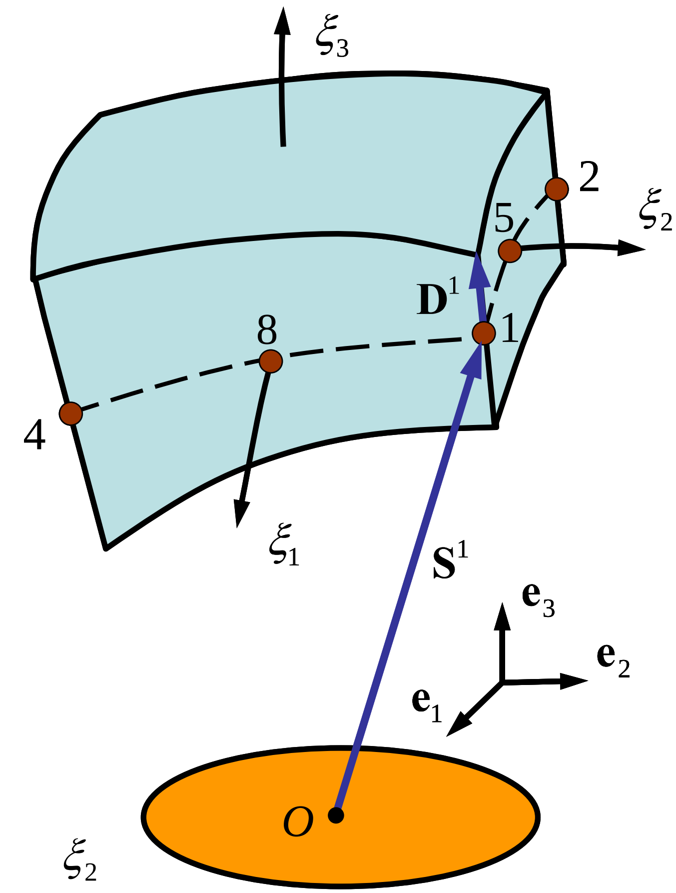


Isoparametric shell

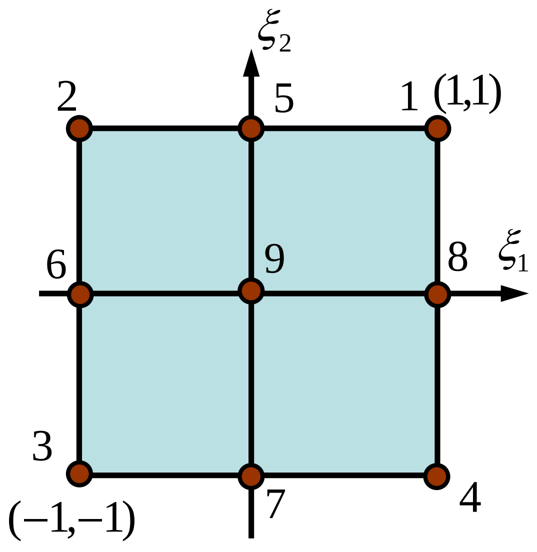
$$\mathbf{X}(\xi_1,\xi_2,\xi_3)=\mathbf{S}(\xi_1,\xi_2)+\varsigma_o(\xi_1,\xi_2,\xi_3)\mathbf{D}(\xi_1,\xi_2)$$

$$\boxed{\begin{aligned}\mathbf{S}(\xi_1,\xi_2)&=\sum_{p=1}^{nnpe}\mathbf{S}^p N_p(\xi_1,\xi_2)\\ \mathbf{D}(\xi_1,\xi_2)&=\sum_{p=1}^{nnpe}\mathbf{D}^p N_p(\xi_1,\xi_2)\\ \varsigma_o(\xi_1,\xi_2,\xi_3)&=\sum_{p=1}^{nnpe}\varsigma_o^p(\xi_3)N_p(\xi_1,\xi_2)\end{aligned}}$$

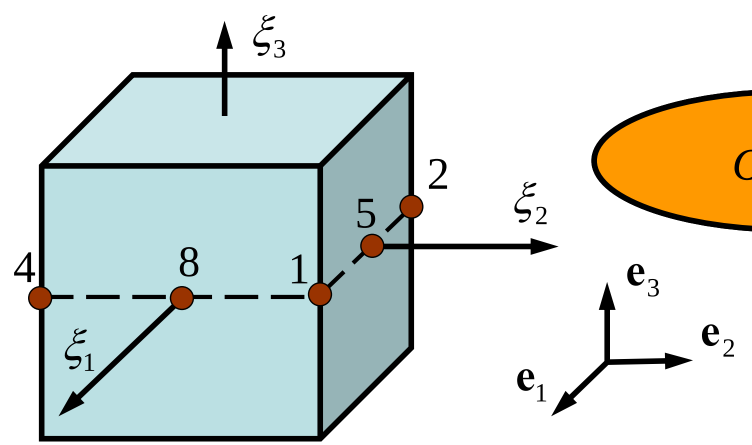
Initial Configuration



Isoparametric 2-D



Extruded 3-D

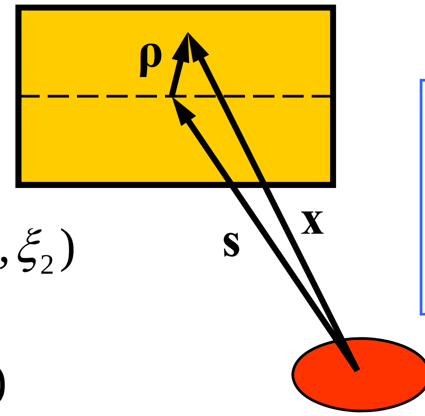


Jacobians

$$\mathbf{J} = \frac{\partial \mathbf{x}}{\partial \xi_i} \otimes \mathbf{e}_i$$

$$\mathbf{J}_o = \frac{\partial \mathbf{X}}{\partial \xi_i} \otimes \mathbf{e}_i$$

$$\mathbf{x}(\xi_1,\xi_2,\xi_3) = \mathbf{s}(\xi_1,\xi_2) + \boldsymbol{\rho}(\xi_1,\xi_2,\xi_3)$$



$$\mathbf{x}(\xi_1,\xi_2,\xi_3) = \mathbf{s}(\xi_1,\xi_2) + \zeta(\xi_1,\xi_2,\xi_3)\mathbf{d}(\xi_1,\xi_2)$$

$$\Delta \mathbf{s}^p = \mathbf{s}^p - \mathbf{S}^p$$
$$\Delta \mathbf{d}^p = \mathbf{d}^p - \mathbf{D}^p$$
$$\Delta \zeta^p = \zeta^p - \zeta_o^p$$

$$\mathbf{x}(\xi_1,\xi_2,\xi_3) = \mathbf{s}^p N_p(\xi_1,\xi_2) + \zeta^p(\xi_3) N_p(\xi_1,\xi_2)\mathbf{d}^q N_q(\xi_1,\xi_2)$$

$$\frac{\partial \mathbf{x}}{\partial \xi_i} = \mathbf{s}^p \frac{\partial N_p(\xi_1,\xi_2)}{\partial \xi_i} + \frac{\partial \zeta^p(\xi_3)}{\partial \xi_i} N_p(\xi_1,\xi_2)\mathbf{d}^q N_q(\xi_1,\xi_2)$$
$$+ \zeta^p(\xi_3)\frac{\partial N_p(\xi_1,\xi_2)}{\partial \xi_i}\mathbf{d}^q N_q(\xi_1,\xi_2) + \zeta^p(\xi_3) N_p(\xi_1,\xi_2)\mathbf{d}^q \frac{\partial N_q(\xi_1,\xi_2)}{\partial \xi_i}$$

Thickness unknowns

$$\zeta^p(\xi_3) = \zeta_o^p(\xi_3) + \sum_{i=1}^{nif} a_i^p f_i(\xi_3)$$

Thickness base functions

$$\frac{\partial \mathbf{X}}{\partial \xi_i} = \mathbf{S}^p \frac{\partial N_p(\xi_1,\xi_2)}{\partial \xi_i} + \frac{\partial \zeta_o^p(\xi_3)}{\partial \xi_i} N_p(\xi_1,\xi_2)\mathbf{D}^q N_q(\xi_1,\xi_2)$$
$$+ \zeta_o^p(\xi_3)\frac{\partial N_p(\xi_1,\xi_2)}{\partial \xi_i}\mathbf{D}^q N_q(\xi_1,\xi_2) + \zeta_o^p(\xi_3) N_p(\xi_1,\xi_2)\mathbf{D}^q \frac{\partial N_q(\xi_1,\xi_2)}{\partial \xi_i}$$

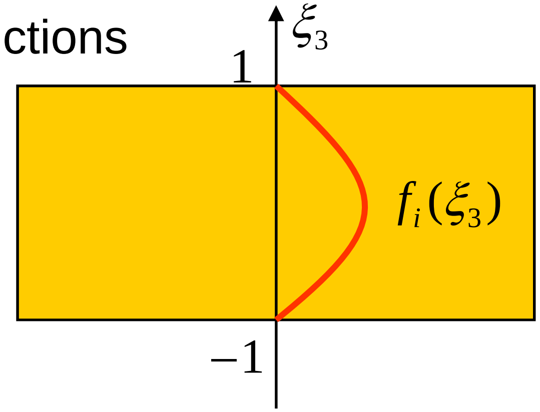
$$\zeta_o^p(\xi_3) = \xi_3$$

Thickness interpolation

$$\zeta_o^p(\xi_3) = \xi_3$$

Number of interpolation functions

$$\zeta^p(\xi_3) = \zeta_o^p(\xi_3) + \sum_{i=1}^{nif} a_i^p f_i(\xi_3)$$

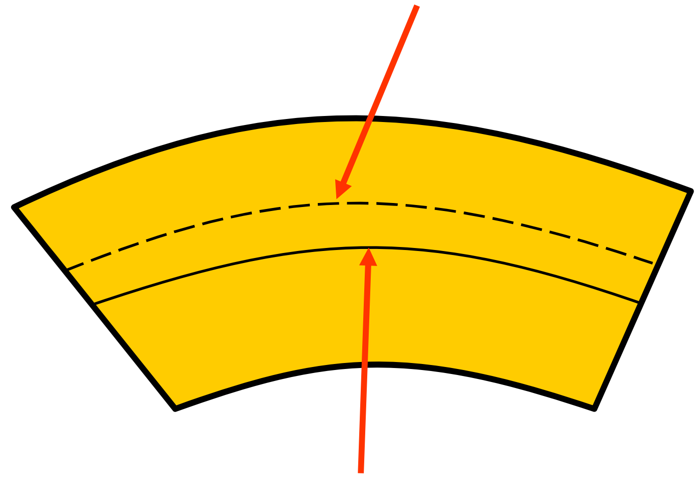


Legendre polynomials:

$$P_o(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1),$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x), \quad P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \ldots$$

$$f_i(\xi_3) = \begin{cases} P_{i+1}(\xi_3) - P_o(\xi_3) & i \text{ is odd} \\ P_{i+1}(\xi_3) - P_1(\xi_3) & i \text{ is even} \end{cases}$$



One term approximation:

$$\Delta\zeta_1^p(\xi_3) = a_1^p f_1(\xi_3) = \Delta\zeta_1^p(0)(1 - \xi_3^2)$$

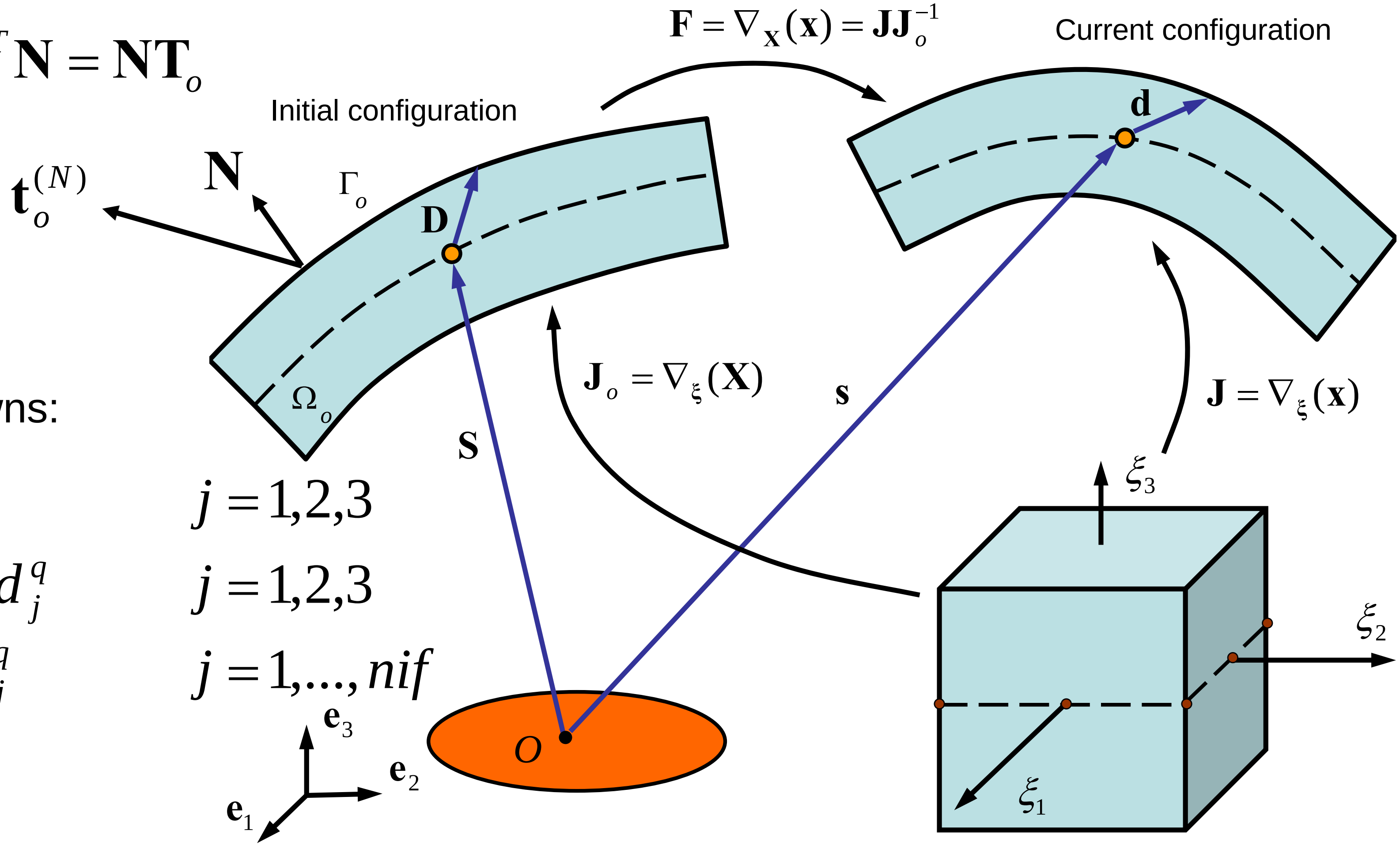
FEM formulation

Residual for FEM:

Nominal stress $\mathbf{T}_o = J\mathbf{F}^{-1}\mathbf{T}$

$$\boxed{R = \int_{\Gamma_o} \bar{\mathbf{u}} \circ \mathbf{t}_o^{(N)} d\Gamma_o - \int_{\Omega_o} \nabla_{\mathbf{X}}(\bar{\mathbf{u}}) : \mathbf{T}_o^T d\Omega_o + \int_{\Omega_o} \rho_o \bar{\mathbf{u}} \circ \mathbf{b} d\Omega_o - \int_{\Omega_o} \rho_o \bar{\mathbf{u}} \circ \mathbf{a} d\Omega_o}$$

$$\mathbf{t}_o^{(N)} = \mathbf{T}_o^T \mathbf{N} = \mathbf{N}\mathbf{T}_o$$



Nodal unknowns:

$$\begin{cases} U_{qj} = \Delta s_j^q & j = 1,2,3 \\ U_{q(j+3)} = \Delta d_j^q & j = 1,2,3 \\ U_{q(j+6)} = a_j^q & j = 1,\ldots,nif \end{cases}$$

Nonlinear elasticity

Deformation gradient: (general)

$$\mathbf{F}_{new} = \mathbf{F}_{old} + \dot{\mathbf{F}}\Delta t$$

$$\mathbf{F}_{new} = \mathbf{F}_{old} + \frac{\partial \mathbf{F}}{\partial U_{qj}}\dot{U}_{qj}\Delta t = \mathbf{F}_{old} + \frac{\partial \mathbf{F}}{\partial U_{qj}}\Delta U_{qj}$$

$$\frac{\partial \mathbf{F}}{\partial U_{qj}} = \frac{\partial \mathbf{J}\mathbf{J}_o^{-1}}{\partial U_{qj}} = \frac{\partial \mathbf{J}}{\partial U_{qj}}\mathbf{J}_o^{-1} = \frac{\partial \mathbf{g}_i \otimes \mathbf{e}_i}{\partial U_{qj}}\mathbf{J}_o^{-1}$$

$$\boxed{\frac{\partial \mathbf{F}}{\partial U_{qj}} = \frac{\partial \mathbf{g}_i}{\partial U_{qj}} \otimes \mathbf{e}_i \mathbf{J}_o^{-1}} \qquad \mathbf{g}_i = \frac{\partial \mathbf{x}}{\partial \xi_i}$$

Nominal stress: (nonlinear elastic)

$$\mathbf{T}_o(\mathbf{F}_{new}) = \mathbf{T}_o(\mathbf{F}_{old}) + \dot{\mathbf{T}}\Delta t$$

$$\mathbf{T}_o(\mathbf{F}_{new}) = \mathbf{T}_o(\mathbf{F}_{old}) + \partial_{\mathbf{F}}(\mathbf{T}) : \dot{\mathbf{F}}\Delta t = \mathbf{T}_o(\mathbf{F}_{old}) + \partial_{\mathbf{F}}(\mathbf{T}_o) : \frac{\partial \mathbf{F}}{\partial U_{qj}}\Delta U_{qj}$$

From shell kinematics

$$\boxed{\mathbf{T}_o(\mathbf{F}_{new}) = \mathbf{T}_o(\mathbf{F}_{old}) + \mathbf{E_o} : \frac{\partial \mathbf{F}}{\partial U_{qj}}\Delta U_{qj}}$$

Material specific

$$\begin{cases} U_{qj} = \Delta s_j^q & j = 1,2,3 \\ U_{q(j+3)} = \Delta d_j^q & j = 1,2,3 \\ U_{q(j+6)} = a_j^q & j = 1,\ldots,nif \end{cases}$$

Tangent modulus of nominal stress: $\mathbf{E_o} = \partial_{\mathbf{F}}(\mathbf{T}_o)$

Nonlinear elasticity

Two parameter nonlinear-elastic material model:

Nominal stress:

$$\mathbf{T}_c = det(\mathbf{F})\mathbf{F}^{-1}\mathbf{T} = \bar{J}[\kappa(\bar{J}-1)\mathbf{F}^{-1} + G\frac{1}{\bar{J}^{5/3}}(\mathbf{F}^T - \frac{I}{3}\mathbf{F}^{-1})]$$

Tangent modulus:

$$E_{ijmn} = \kappa\bar{J}(2\bar{J}-1)F_{ij}^{-1}F_{nm}^{-1} - \frac{2}{3}\frac{G}{\bar{J}^{2/3}}F_{nm}^{-1}(F_{ji} - \frac{I}{3}F_{ij}^{-1})$$
$$- \kappa\bar{J}(\bar{J}-1)F_{im}^{-1}F_{nj}^{-1} + \frac{G}{\bar{J}^{2/3}}(\delta_{in}\delta_{jm} - \frac{2}{3}F_{mn}F_{ij}^{-1} + \frac{I}{3}F_{im}^{-1}F_{nj}^{-1})$$

Material parameters: Shear and bulk modulus matched to linear response

Examples from Nonlinear Elasticity

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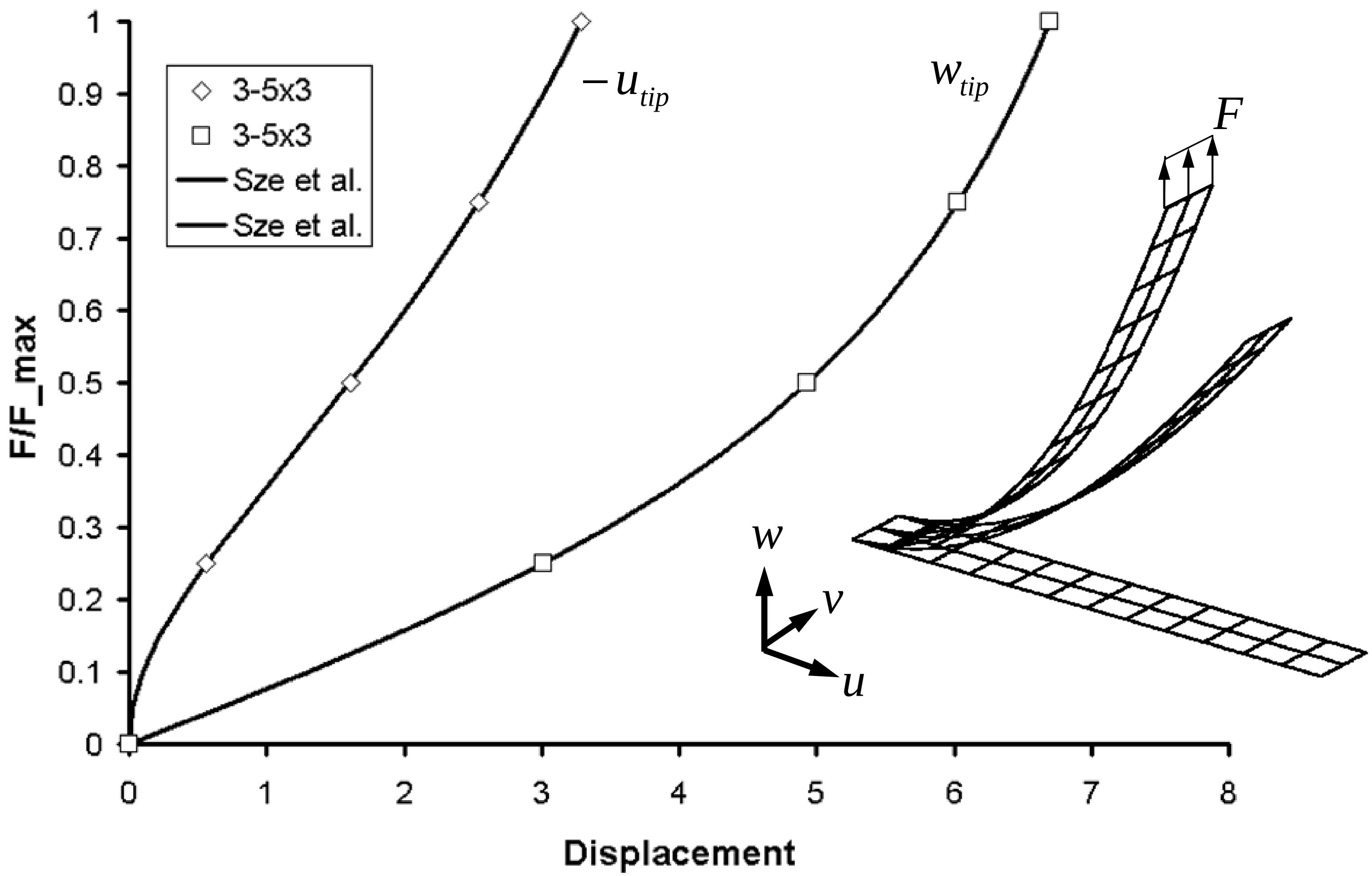
Cantilevered beam with distribute end load

4 Load steps
41 Newton iterations

$$E = 1.2 \times 10^6 \quad \nu = 0.0 \quad F_{\max} = 4.0$$

$$L = 10.0 \quad h = 0.1 \quad b = 1.0$$



Examples fromNonlinear Elasticity

Cantilevered beam with distribute end moment

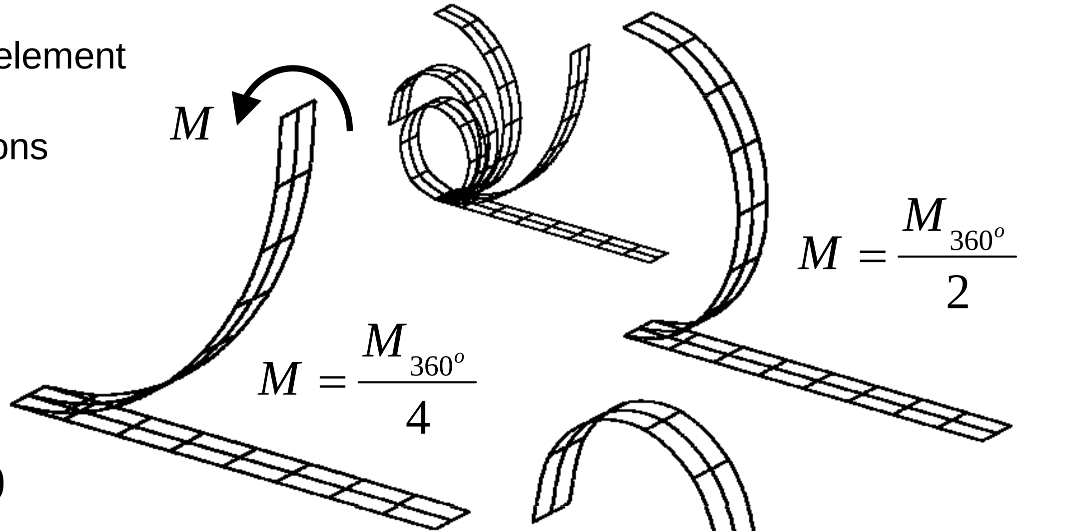
Single (9x3)-node element
16 loading steps
175 Newton iterations

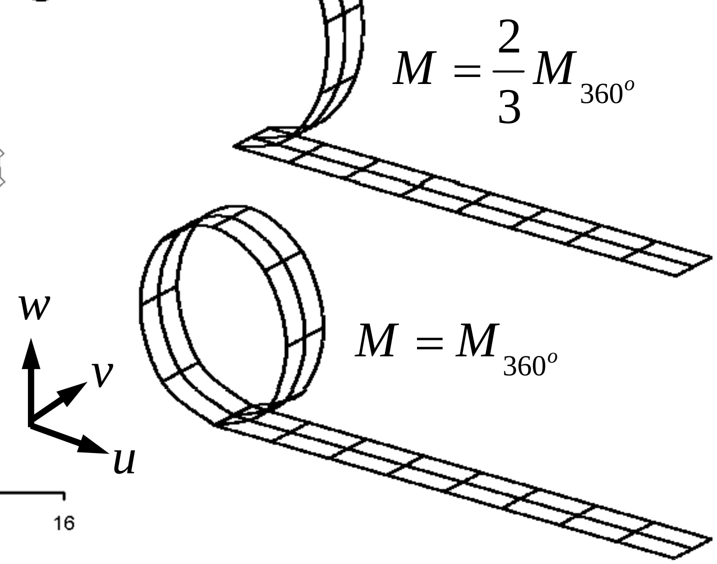
$$E = 1.2 \times 10^6$$

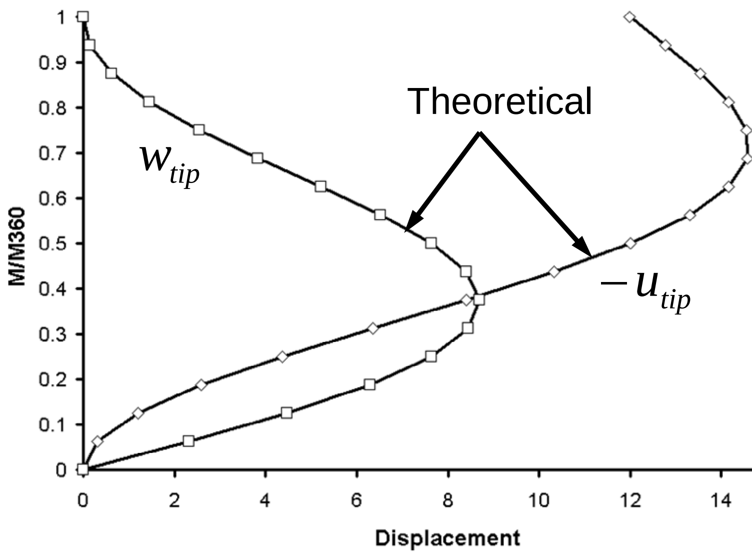
$$\nu = 0.0$$

$$L = 12.0$$

$$h = 0.1 \quad b = 1.0$$

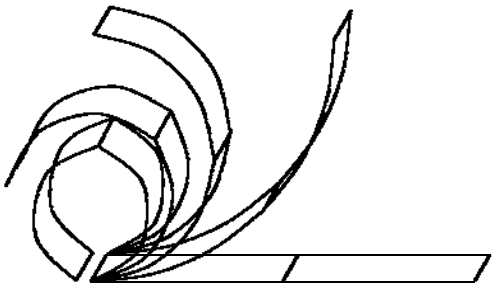




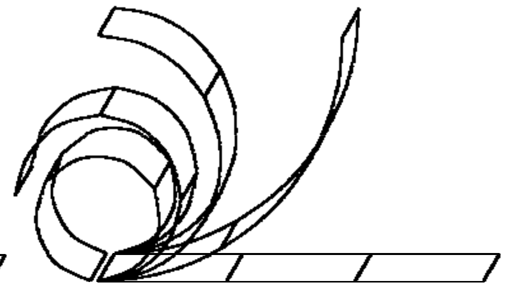


Examples from Nonlinear Elasticity

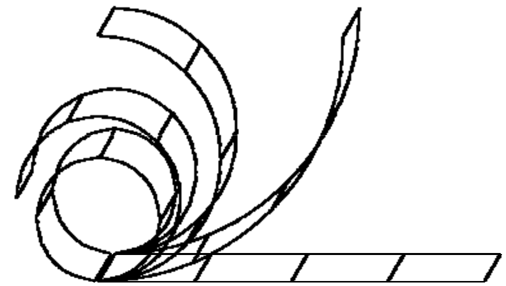
Cantilevered beam with distribute end moment



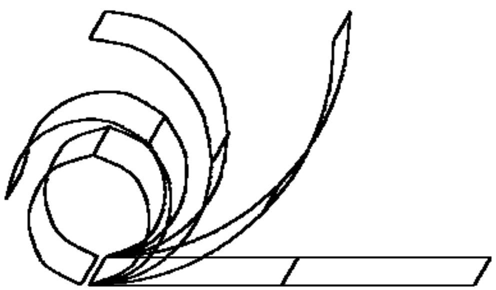
(a) 2-(4x3)-360^o-16LINC-187NI



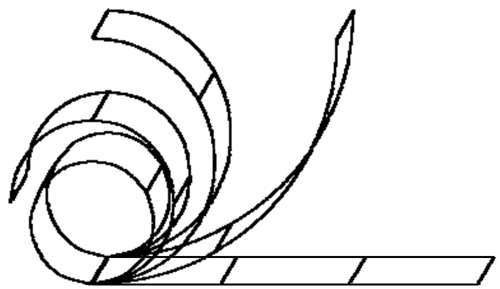
(b) 3-(4x3)-360^o-16LINC-147NI



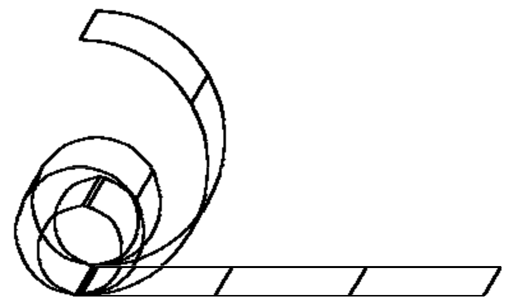
(c) 4-(4x3)-360^o-16LINC-147NI



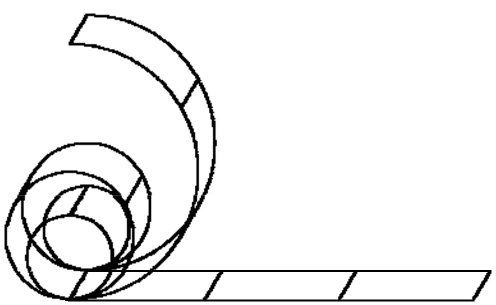
(d) 2-(5x3)-360^o-16LINC-150NI



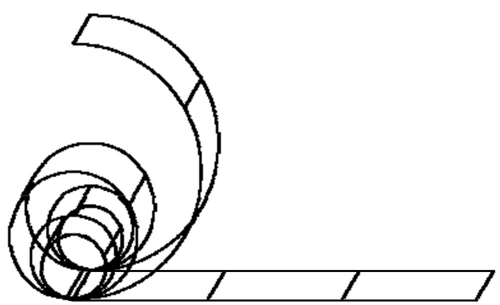
(e) 3-(5x3)-360^o-16LINC-150NI



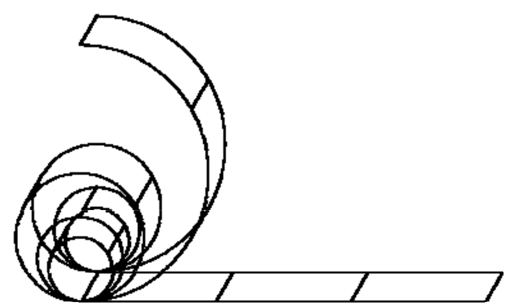
(f) 3-(5x3)-540^o-24LINC-224NI



(g) 3-(6x3)-540^o-24LINC-221NI

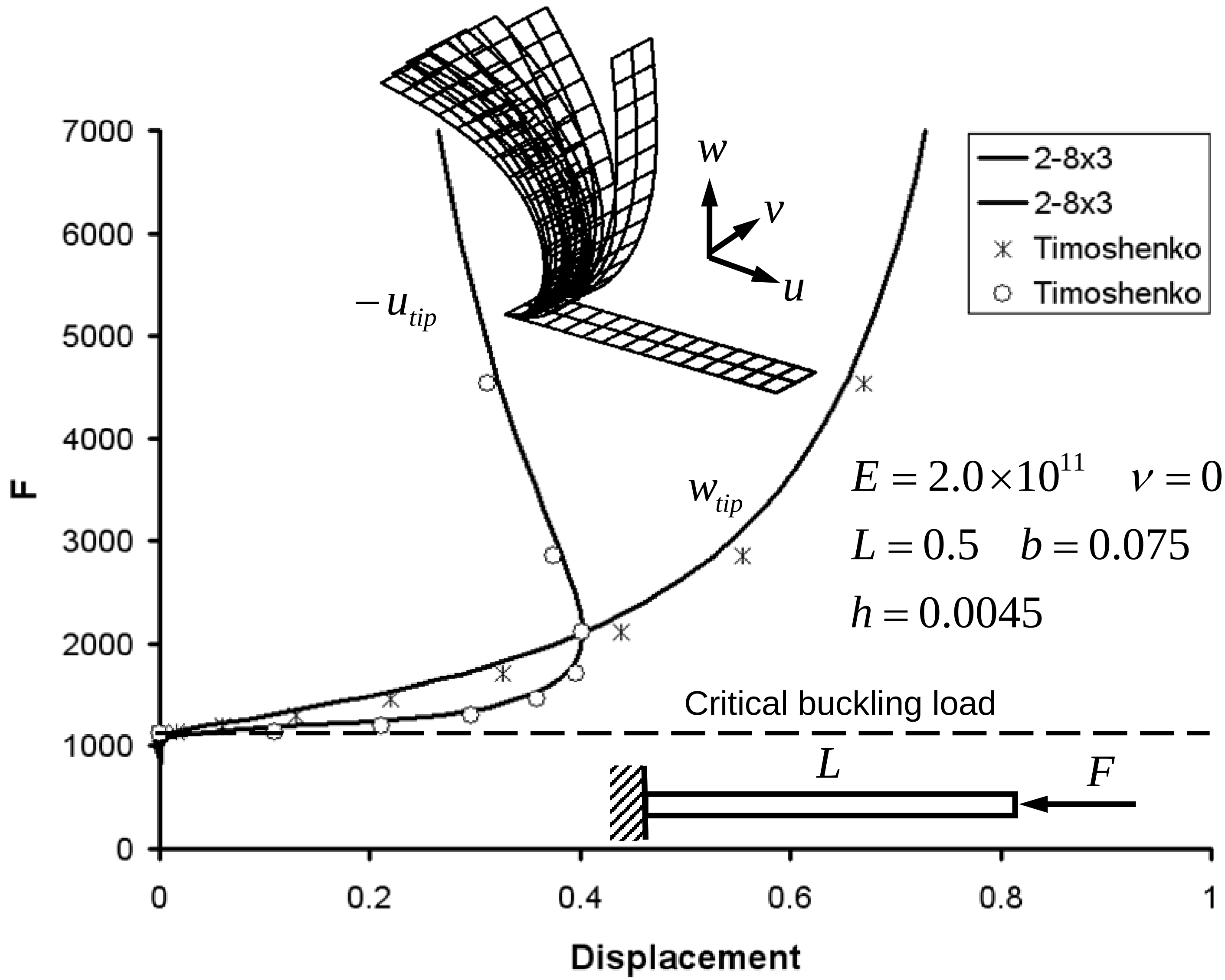


(h) 3-(6x3)-720^o-32LINC-309NI



(i) 3-(7x3)-720^o-32LINC-329NI

Buckling in compression of a column



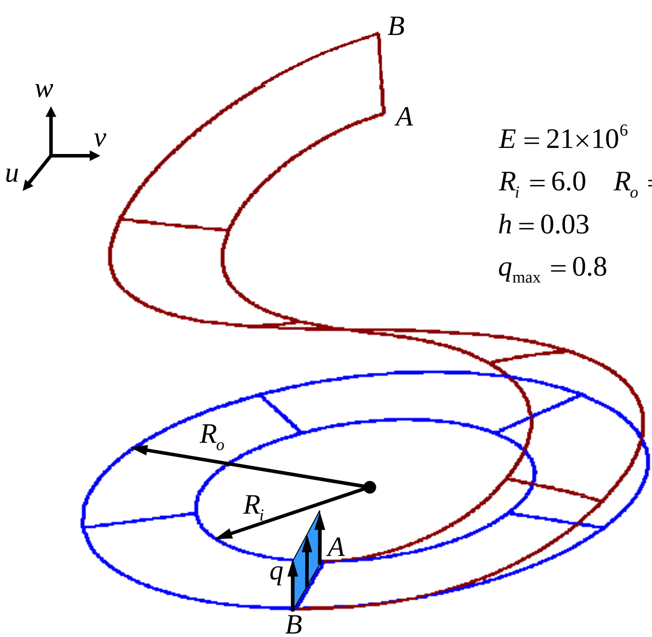
$$E = 2.0\times10^{11} \quad \nu = 0.3$$

$$L = 0.5 \quad b = 0.075$$

$$h = 0.0045$$

Examples from Nonlinear Elasticity

Transverse loading of an annular ring

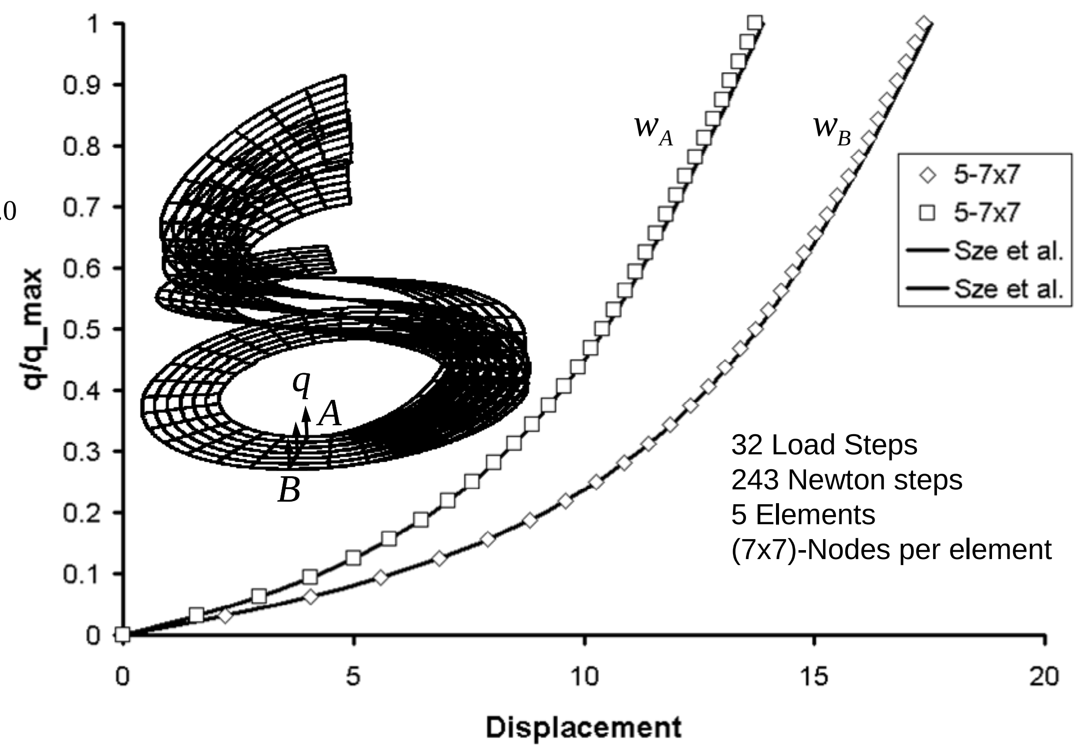


$E = 21\times10^6 \quad \nu = 0.0$

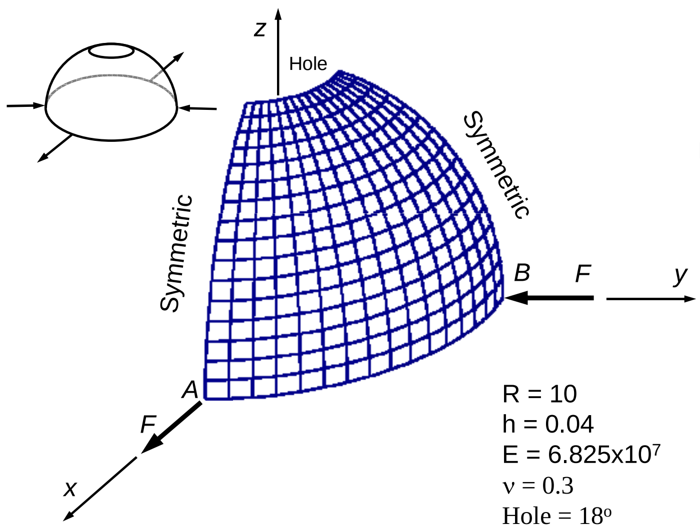
$R_i = 6.0 \quad R_o = 10.0$

$h = 0.03$

$q_{max} = 0.8$



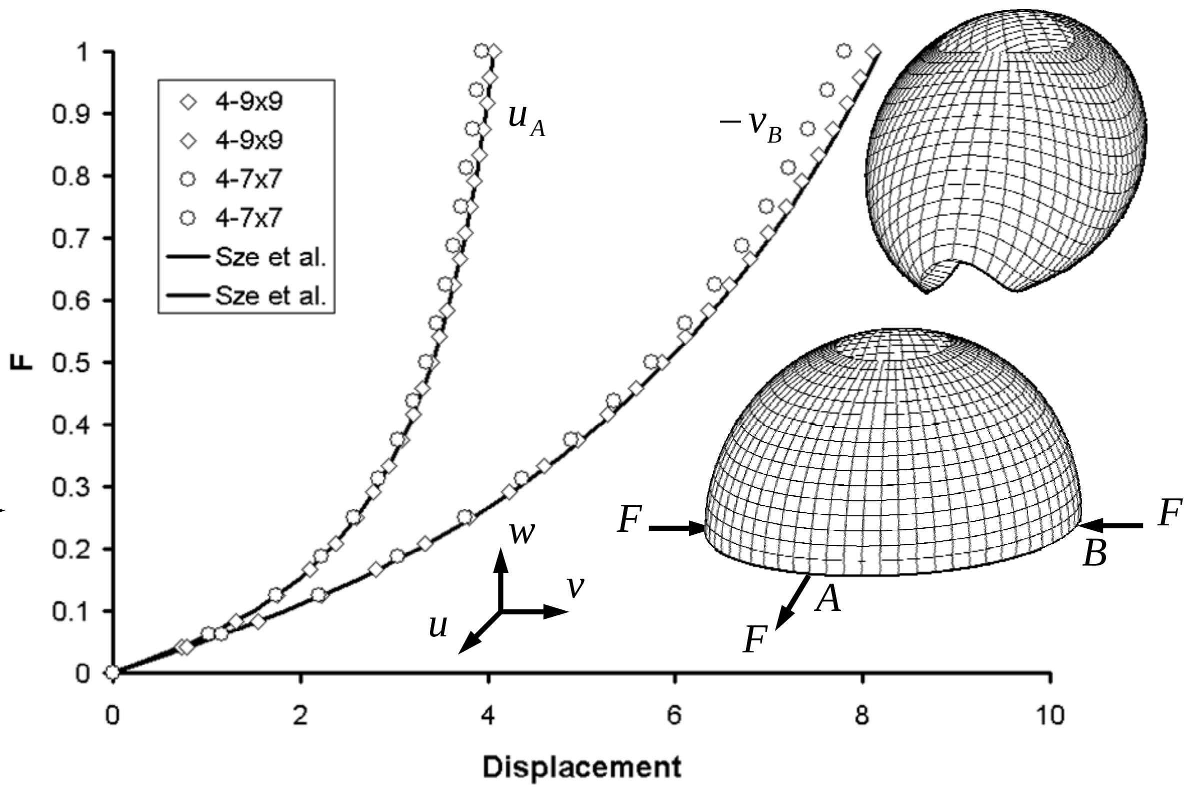
Pinching of hemispherical shell with hole



R = 10
h = 0.04
E = 6.825x10^7
ν = 0.3
Hole = 18°

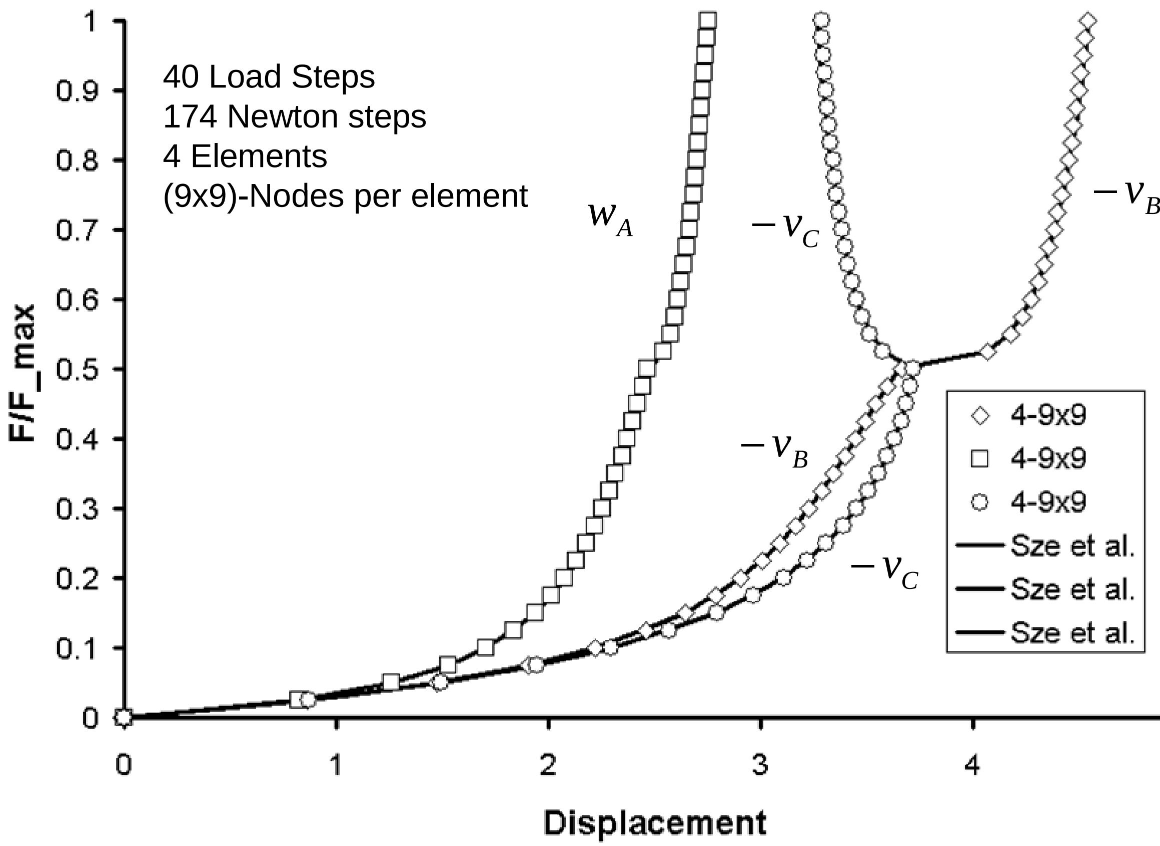
4 Elements
7x7 or 9x9 Node Elements

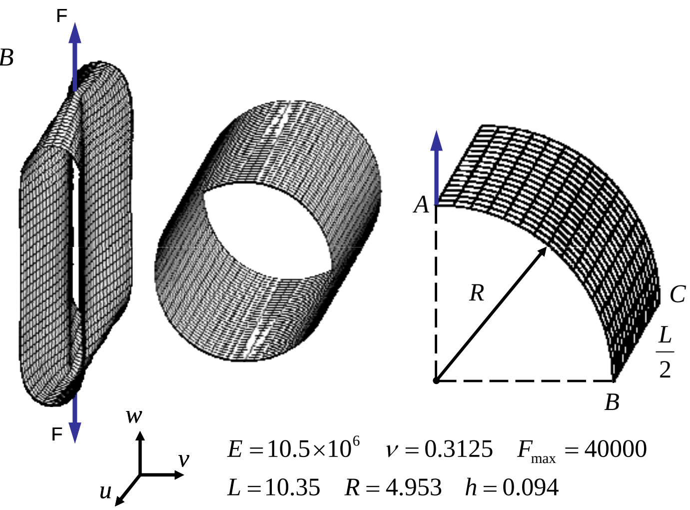
7x7 Element: 16 Load steps, 127 Newton Iterations
9x9 Element: 24 Load steps, 163 Newton Iterations



Examples from Nonlinear Elasticity

Pinching of cylindrical shell with free ends



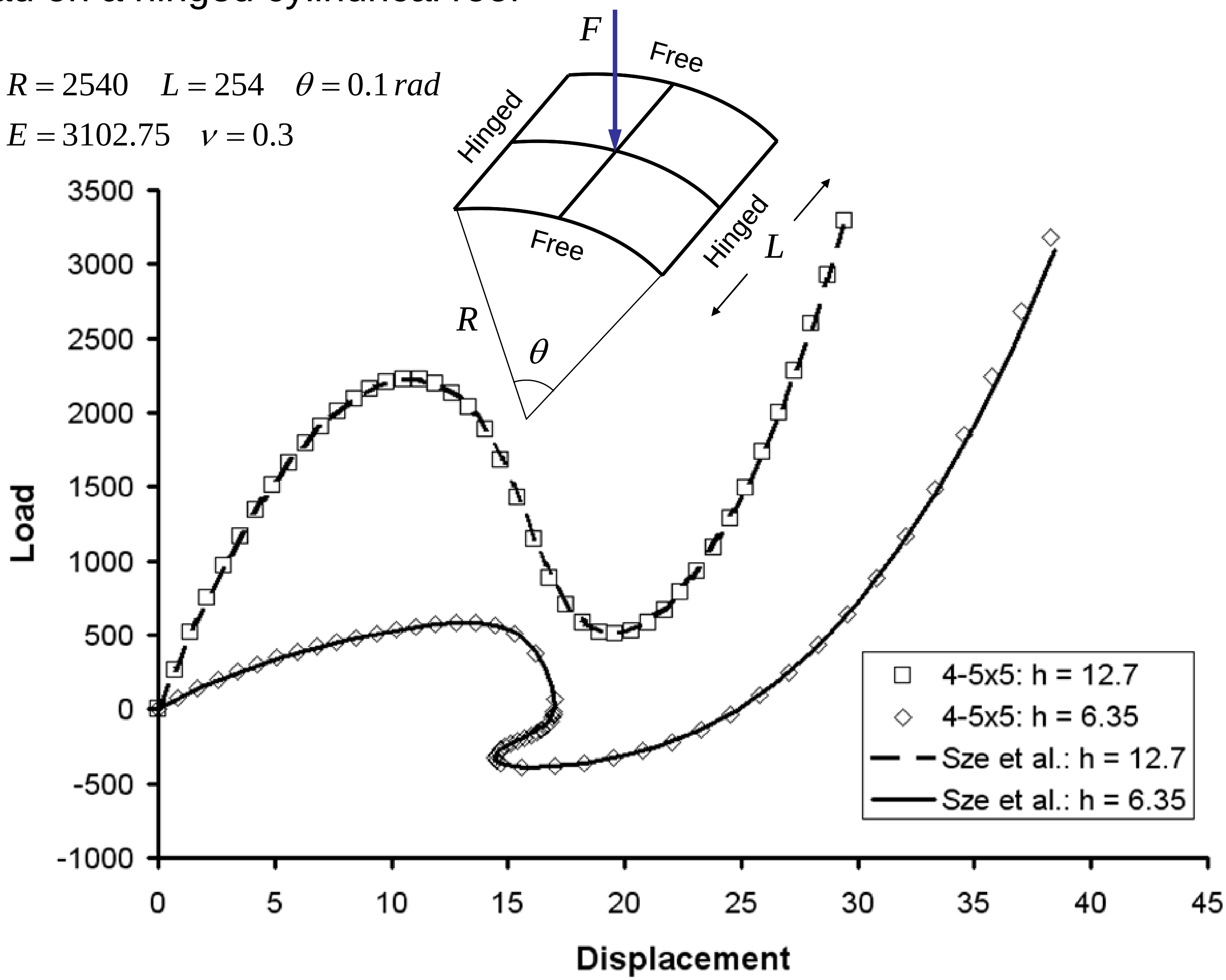


Examples from Nonlinear Elasticity

Point load on a hinged cylindrical roof

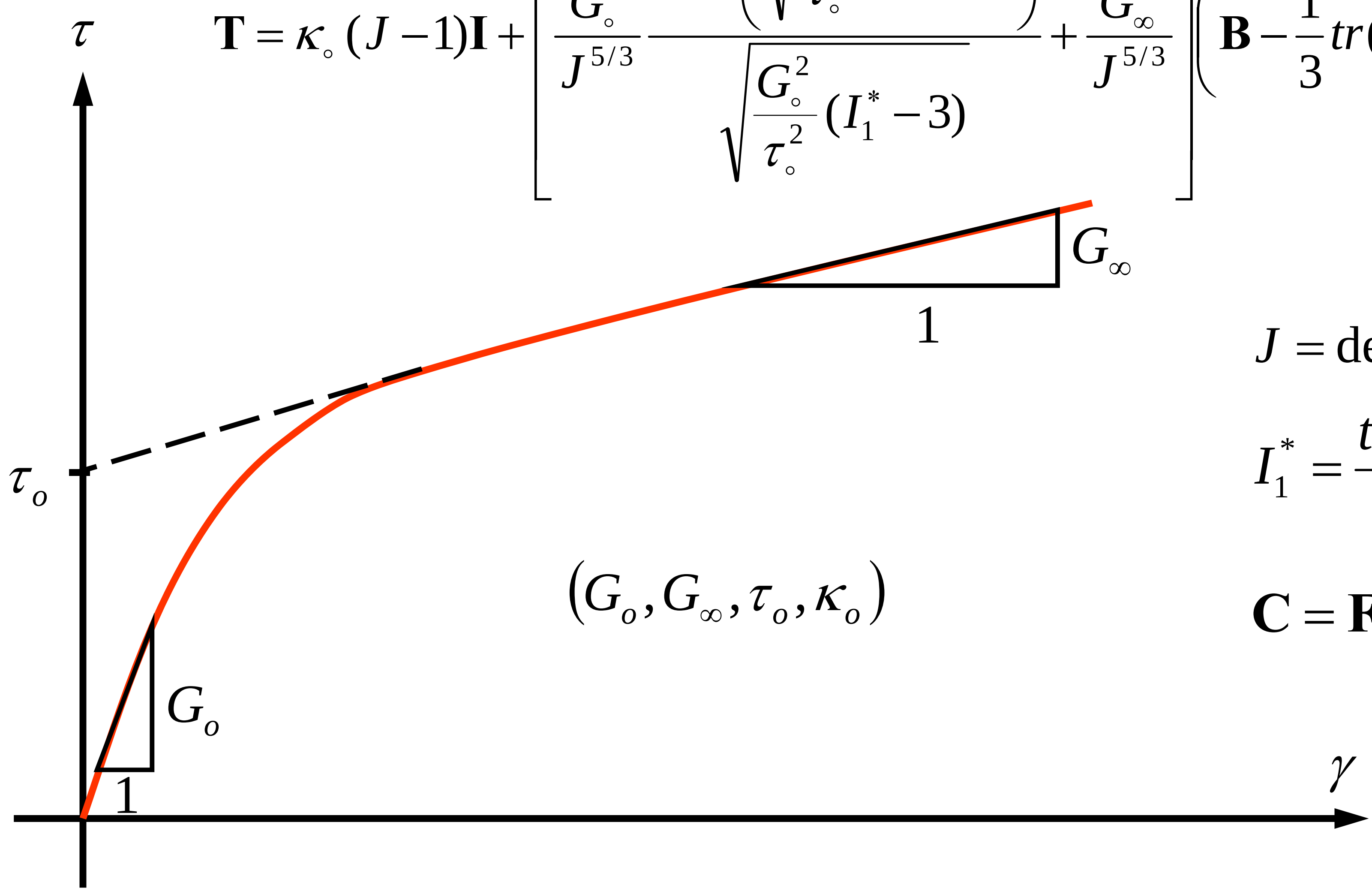
$R = 2540 \quad L = 254 \quad \theta = 0.1\,rad$

$E = 3102.75 \quad \nu = 0.3$



Four parameter elastic model

$$\mathbf{T} = \kappa_\circ (J-1)\mathbf{I} + \left[\frac{G_\circ}{J^{5/3}} \frac{\tanh\left(\sqrt{\frac{G_\circ^2}{\tau_\circ^2}(I_1^* - 3)}\right)}{\sqrt{\frac{G_\circ^2}{\tau_\circ^2}(I_1^* - 3)}} + \frac{G_\infty}{J^{5/3}} \right] \left(\mathbf{B} - \frac{1}{3} tr(\mathbf{B})\mathbf{I} \right)$$



$$\left(G_o, G_\infty, \tau_o, \kappa_o\right)$$

$$J = \det(\mathbf{F})$$

$$I_1^* = \frac{tr(\mathbf{C})}{J^{\frac{2}{3}}}$$

$$\mathbf{C} = \mathbf{F}^T \mathbf{F}$$

Large deformation elastic-plastic model

Cauchy stress:

$$\mathbf{T} = \kappa(J^e - 1)\mathbf{I} + G\frac{1}{J^{e\frac{5}{3}}}\left(\mathbf{B}^e - \frac{tr(\mathbf{B}^e)}{3}\mathbf{I}\right)$$

$$\kappa = \frac{E}{3(1-2\nu)}$$

$$G = \frac{E}{2(1+\nu)}$$

Over-stress:

$$\Delta\mathbf{T} = \mathbf{F}^{\mathbf{e}-1}\mathbf{T}\mathbf{F}^{\mathbf{e}} - \mathbf{T}^{\mathbf{b}}$$

$$\mathbf{T}^{\mathbf{b}} = \mathbf{0}$$

Yield function:

$$f = \Delta\mathbf{S} : \Delta\mathbf{S} - \frac{2}{3}\sigma_y{}^2$$

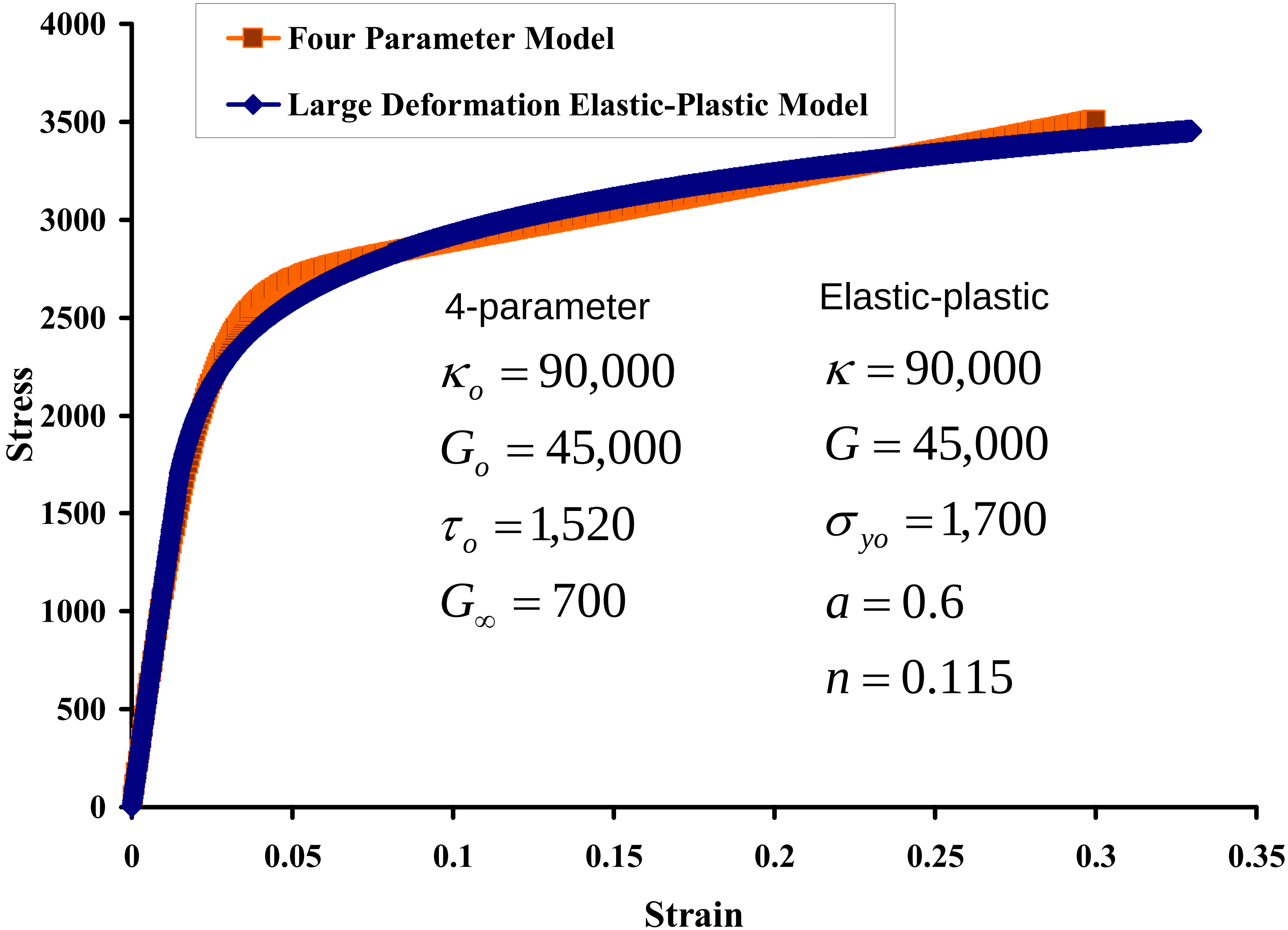
$$\Delta\mathbf{S} = \Delta\mathbf{T} - \frac{tr(\Delta\mathbf{T})}{3}\mathbf{I}$$

Power-law hardening rule:

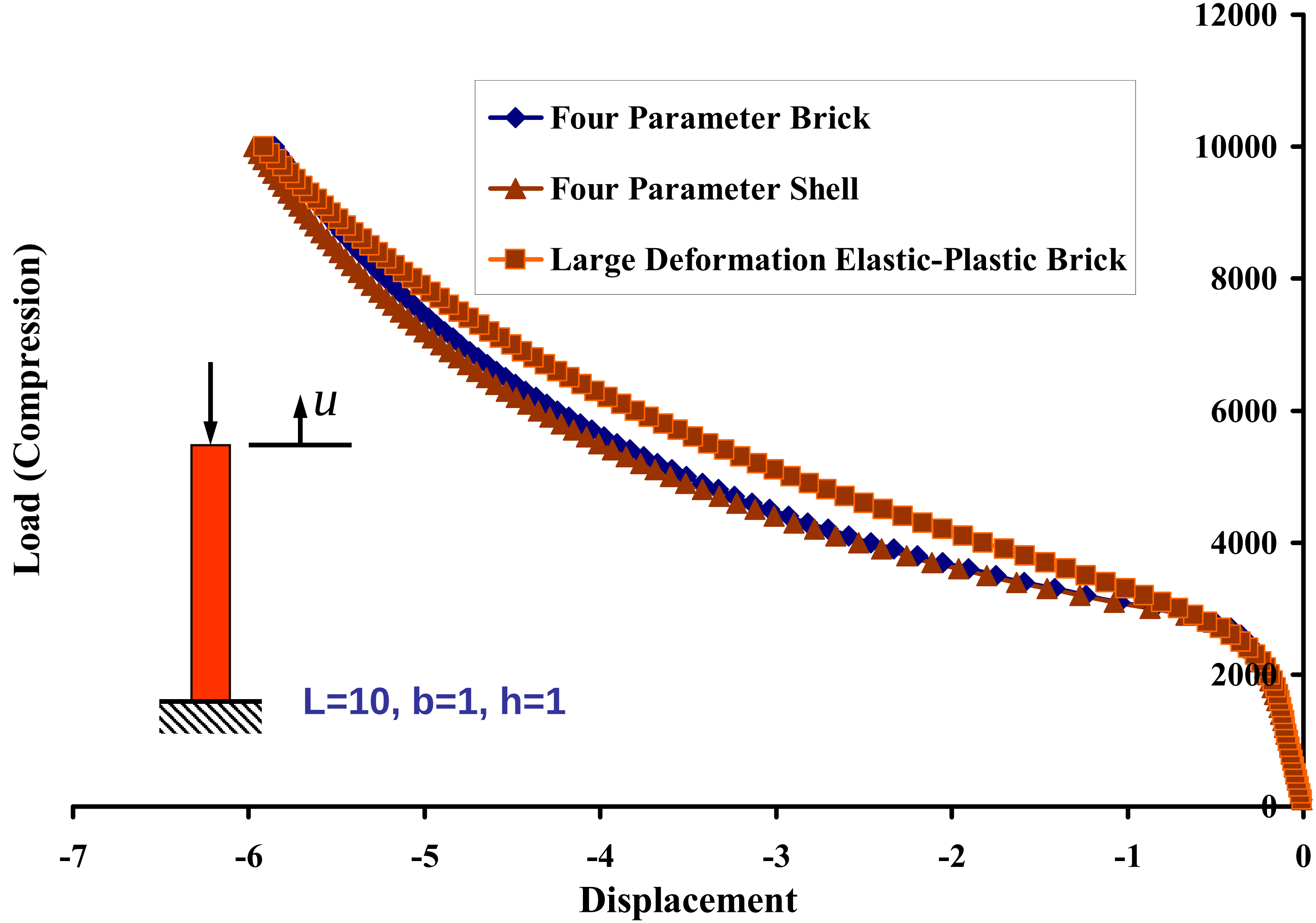
$$\sigma_y = \sigma_{yo}(1 + a\xi)^n$$

$$\dot{\xi} = \Delta\mathbf{T}^T : \mathbf{L}^{\mathbf{p}}$$

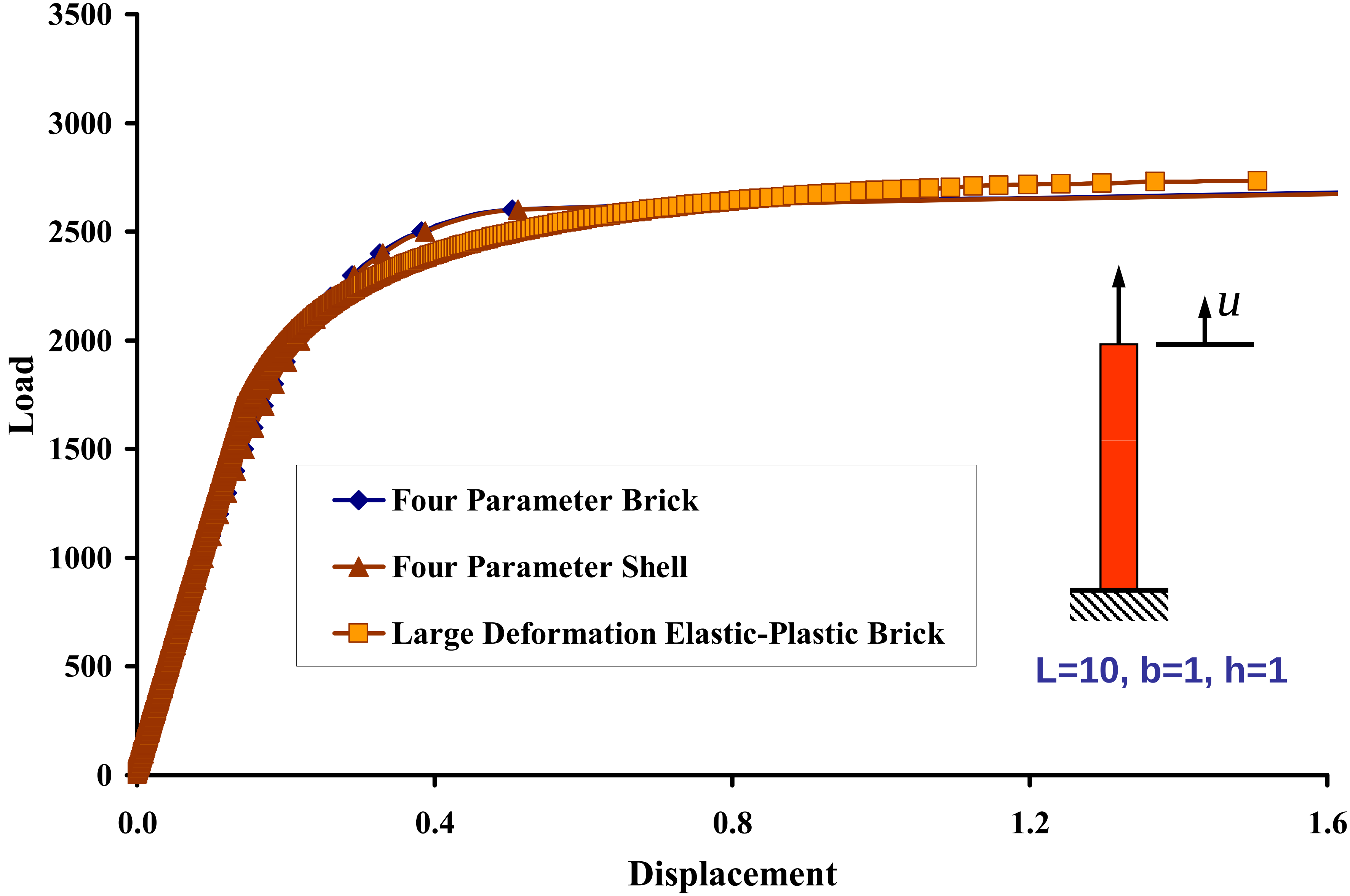
Two models in tension



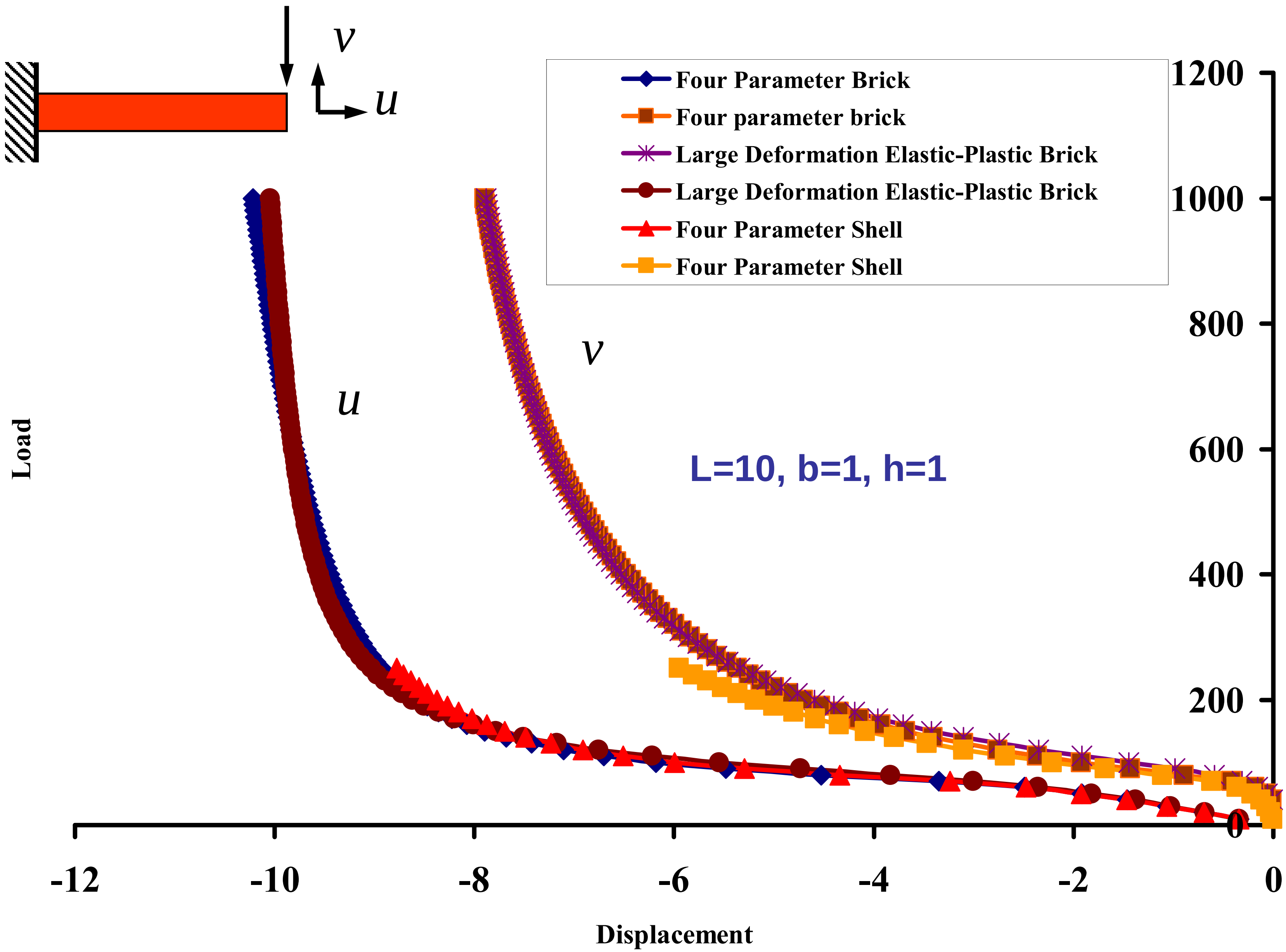
Beam in compression



Beam in tension

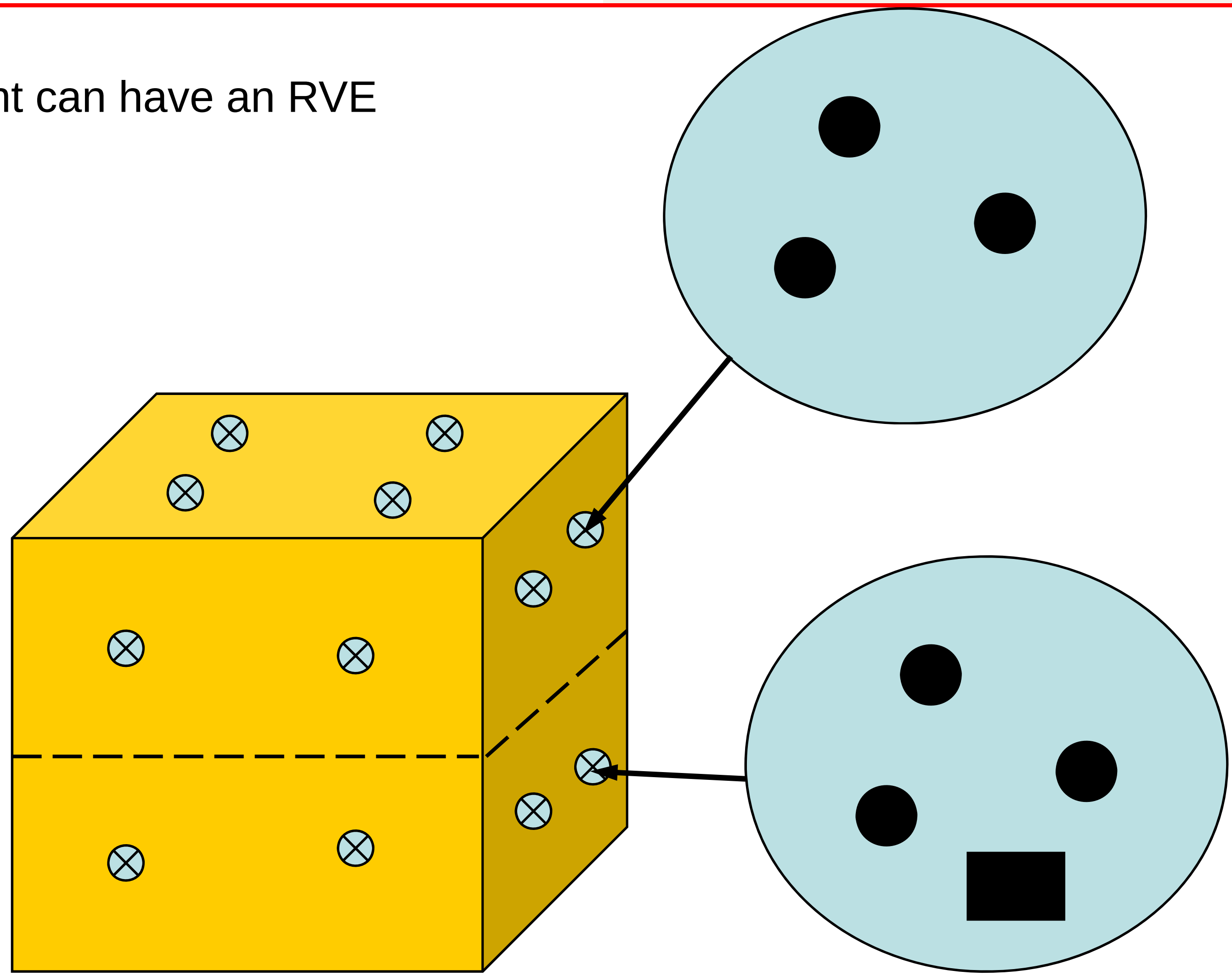


Cantilevered beam

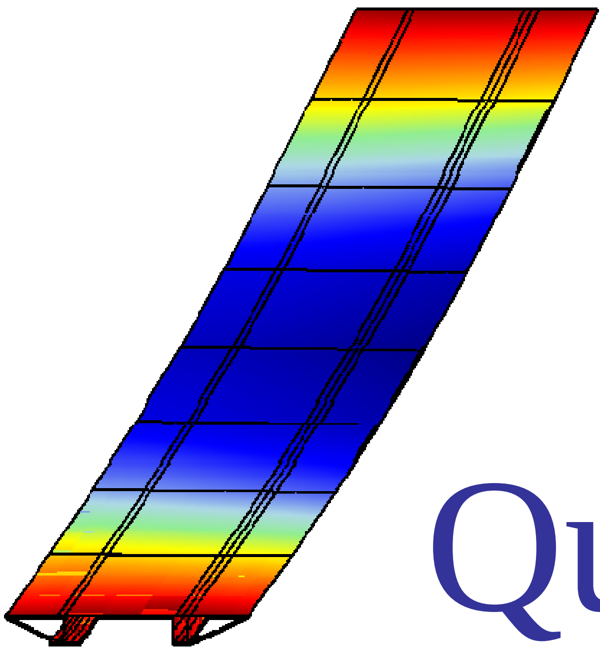


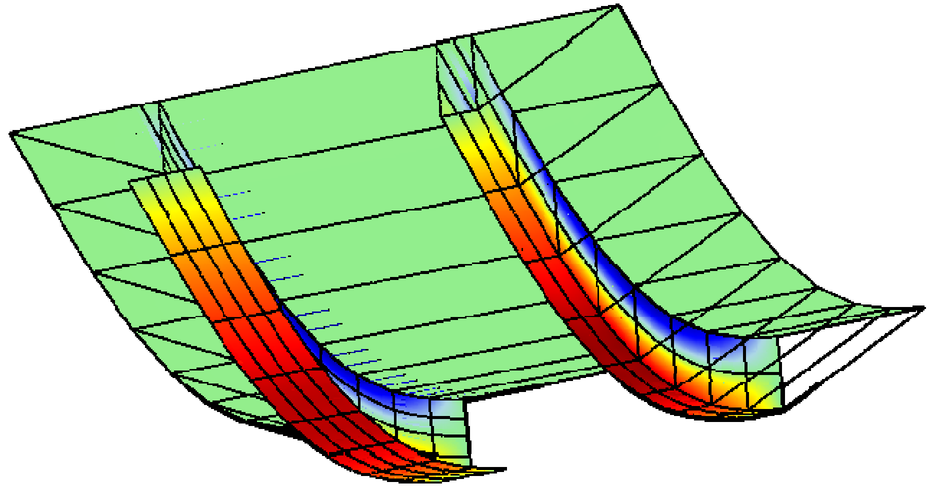
Multi-scale analysis

Each integration point can have an RVE



NaBRO





Questions?